How to Mismatch Consumers*

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Abstract

Should platforms like Amazon or Google match consumers to the best firms? Or might consumers be better off from being mismatched? This paper shows that rational consumers may be better off from being mismatched because it lowers market prices. This paper formalizes and studies this novel price-theoretic rationale for giving consumers fewer, not more, choices.

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1 Introduction

Regulators around the world are increasingly concerned that dominant platforms like Amazon and Google harm consumers by limiting their choices.

For example, in 2017, the European Commission fined Google €2.42 billion for actively demoting rival comparison shopping services in its search results and favoring its own service. This concern was echoed in the US when, in 2019, an investigation by the House Judiciary Committee's antitrust subcommittee found that Google leveraged its search monopoly to promote its inferior vertical offerings, imposing search penalties on third-party providers.

The spotlight is not only on Google. Amazon too has faced its share of scrutiny. A recent FTC lawsuit accused the company of replacing organic search results with paid advertisements and increasing the number of junk advertisements. Further, a 2020 investigation by the European Commission suggested that Amazon's criteria for the Buy Box and Prime might be designed to unduly favor its own retail business and those marketplace sellers that rely on Amazon's logistics.

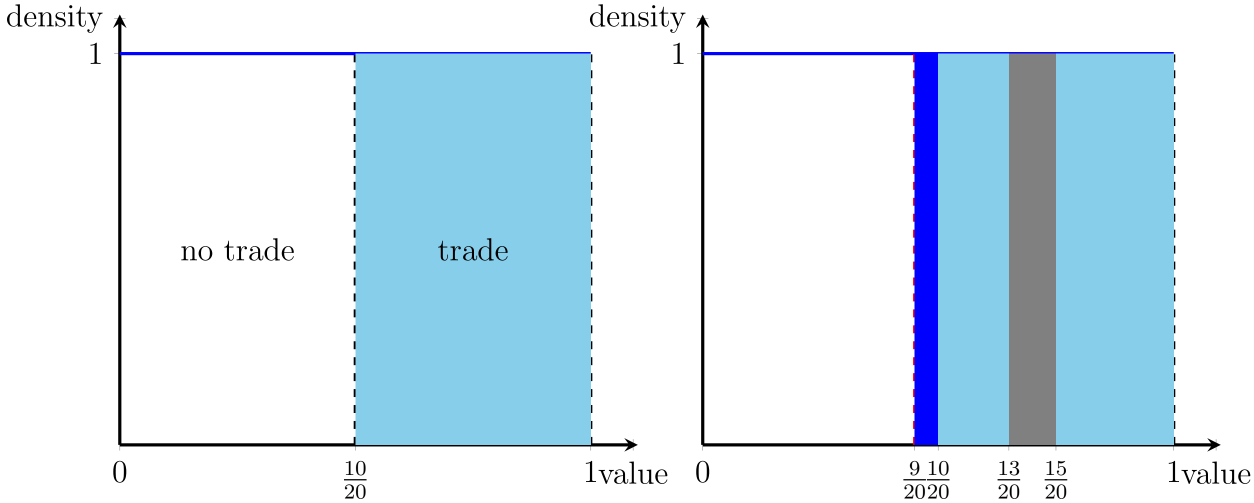
The increasing regulatory scrutiny around giants like Amazon and Google raises important questions about the implications of restricted consumer choice on major platforms. Yet, for such a prominent issue, little is known about the consequences for consumer welfare when dominant platforms curtail access to alternatives. Conventional wisdom suggests that limiting choice is unequivocally detrimental to consumers. But is this always the case? This paper delves into this terrain, posing the question: Could rational consumers, in fact, benefit when their choices are constrained?

Indeed, they can. This paper introduces a novel price-theoretic rationale for restricting consumers' choices with broad applicability. Remarkably, even in the absence of attention or self-control costs, consumers can benefit from a platform artificially reducing their available choices.

To see how, consider a simple example of a monopolistic market. A single firm interacts with a continuum of consumers, whose value is uniformly distributed from 0 to 1. Suppose for simplicity that the cost of production is zero. The platform observes each consumer's valuation and decides which consumers can purchase the good from the firm. Any excluded consumer obtains a payoff of 0.

First, suppose that the platform grants all consumers access to the monopoly, as in Panel (a) in Figure 1. As a result, the monopoly sets a price at $\frac{1}{2}$. Only those consumers who value the product above this price will make a purchase and thus obtain a positive surplus.

Now, look at Panel (b) in the same figure. Here, the platform excludes consumers whose values lie between $\frac{13}{20}$ and $\frac{15}{20}$. By doing so—excluding $\frac{2}{20}$ of consumers—the platform prompts the monopoly to lower its price to $\frac{9}{20}$. The intuition is that by removing these customers, the firm has less to lose (relative to if it accessed all consumers) from lowering prices below $\frac{1}{2}$.[1] As a result of this price reduction: consumers in the two light blue regions benefit from the price drop. Those in the dark blue region can now afford to trade with the monopoly, thereby gaining a positive surplus. On the other hand, the excluded consumers between $\frac{13}{20}$ and $\frac{15}{20}$ are worse off. But taking everything into account, the net gain in consumer surplus is strictly positive.[2]



(a) All consumers have access.

(b) Consumers in the gray band are excluded.

Figure 1: Excluding Consumers May Raise Consumer Surplus

The above argument shows how limiting access makes consumers better off. What

[1] Why this price reduction? When the aforementioned set of consumers is excluded, the monopoly has no incentives to set prices within that range. Breaking it down: For price p under $\frac{13}{20}$, the monopoly's profit would be $p(1-p-\frac{2}{20})$, taking into account the excluded $\frac{2}{20}$ consumers. This function peaks at $\frac{9}{20}$ with a maximum $\frac{81}{400}$. For prices above $\frac{15}{20}$, the optimal profit is $\frac{15}{20}(1-\frac{15}{20})=\frac{75}{400}$, which is realized at precisely $\frac{15}{20}$.

[2] The increase in consumer surplus is $(\frac{1}{2}-\frac{2}{20})\frac{1}{20}+\frac{1}{20}(\frac{9.5}{20}-\frac{9}{20})$. The decrease, due to excluding certain consumers, is $\frac{2}{20}(\frac{14}{20}-\frac{1}{2})$. The net gain in consumer surplus ends up being $\frac{1}{20}\frac{0.5}{20}$. In fact, in this uniform example, for each cutoff $c \in (\frac{1}{2},\frac{3}{4})$, there exists sufficiently small ε so that by excluding consumers in interval $[c, c+\varepsilon]$, the platform can induce a strictly lower monopoly price and a strictly larger consumer surplus.

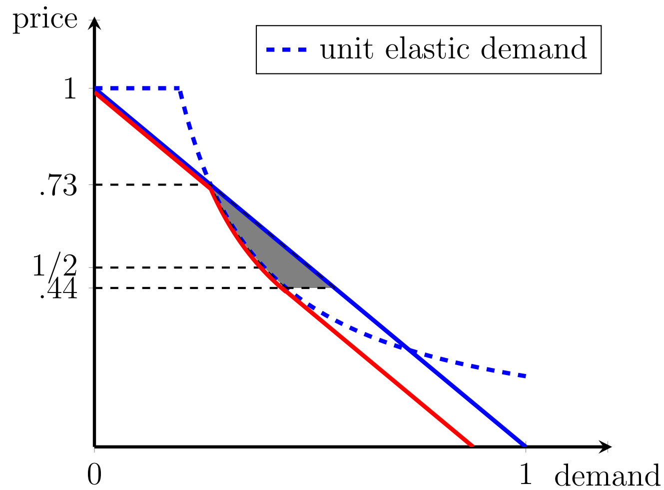


Figure 2: Consumer-Optimal Access Policy

is the best consumer access policy? The consumer-optimal access policy excludes intermediate-value consumers in the interval $[.44, .73]$. However, instead of excluding those consumers with probability one, the consumer-optimal access policy is stochastic and the conditional excluding probability of each consumer in this interval is chosen so that the monopoly is willing to reduce its price to 0.44 and is exactly indifferent between this price and higher prices in the interval. The consequent monopoly demand curve is depicted by the red curve in Figure 2. In this representation, the vertical axis is designated for prices, while demand (or sales) is on the horizontal axis. When all consumers are granted access, the monopoly faces a linear demand curve represented by the blue curve. Notably, within the $[.44, .73]$ interval, the demand curve induced by consumer-optimal access policy aligns with the unit elastic demand characterized by a constant revenue of 0.2. Ultimately, the consumer-optimal access policy increases consumer surplus from $\frac{1}{8}$ to 0.134.

My analysis generalizes the insights from the above uniform example. I show that excluding consumers can strictly improve consumer surplus if value distribution satisfies a few weak conditions. I further characterize the consumer-optimal access policy. The consumer optimal access policy creates a *portion* of a unit-elastic demand curve; in addition, the induced demand curve is pointwise above the demand curve induced by any policy leading to the same level of profit.

I find conditions under which the full access policy is social-optimal; these conditions usually hold.[3] But there are cases where even (utilitarian) efficiency is enhanced by removing types. For example, this happens under the Bounded Pareto distribution with reasonable parameter values. When the full access policy is not social-optimal, I further characterize the social-optimal access policy, which also has the two features mentioned above that belong to the consumer-optimal access policy.

My analysis shows that one can obtain further insights when there are multiple firms. To increase consumer surplus beyond the full access level, the platform excludes consumers with strong preferences. In those words, these consumers have value for their preferred products just above the equilibrium prices but only negligible values for other products. I further show that even if the platform can only exclude at most one firm from each consumer's consideration set, it is still better to mismatch some consumers than to show the full range of choices to each consumer.

It's important to emphasize the broader implications and potential regulatory concerns of this analysis. It provides a price-theoretic rationale for how consumers can benefit when their choices are restricted. This is a nuanced perspective that often isn't highlighted in most discussions about platform behaviors.

However, it's crucial to acknowledge features omitted in my model. I focus on the direct price effects, but there could be additional consumer benefits in scenarios where limited attention plays a role. Conversely, there could be detrimental consequences if a platform's choice restrictions discourage market entry or stifle innovation. Moreover, my model does not necessarily depict how profit-maximizing platforms operate in the real world, given that I do not model platform fees, advertising revenue, and other relevant revenue streams.[4] Yet, the underlying message is clear: due to fundamental market forces, it isn't always the case that consumers are better off from having more choices. Platforms might indeed entice a larger user base precisely by curbing their available options.

Some ingredients of my analysis may be reminiscent of recent advances in information design. Condorelli and Szentes (2020) show that by stochastically destroying her value, a consumer can be better off in a hold-up problem. Roesler and Szentes (2017) also shows that by generating a unit-elastic demand curve through one's information, a consumer may be better off. An important difference between my analysis and these

[3]An access policy is social-optimal if it maximizes total surplus among all policies.

[4]Instead, Bonatti, Bergemann, and Wu (2023) model the sale of advertising positions on digital platforms and its impact on product prices.

prior papers is that the only instrument here is that of matching. Through matching, the consumer-optimal policy creates a *portion* of a unit-elastic demand curve. Moreover, this paper studies both monopolistic and oligopolistic settings, unlike this prior work that focuses on interactions with a single seller.[5]

While much of the existing literature focuses on market segmentation or price discrimination, my paper highlights a new avenue through which consumers might derive benefits from platforms. In a seminal paper, Bergemann, Brooks, and Morris (2015) demonstrates that in a monopoly market, consumers can benefit from market segmentation and further characterize the consumer-optimal market segmentation. Elliott, Galeotti, Koh, and Li (2022a) study an oligopoly market, showing how to optimally segment the market to amplify price competition between firms. My research encompasses both monopoly and oligopoly markets, showing that even without market segmentation, consumers can still benefit from a platform that artificially restricts their access to firms. Elliott, Galeotti, Koh, and Li (2022b) comprehensively characterizes the power a platform holds when controlling both market segmentation and consumers' access to firms.

Focusing on different forces, Condorelli and Szentes (2022a) study a vertically differentiated market in which a continuum of sellers with heterogeneous and observable product qualities are matched with buyers who have a private value for quality. They show the buyer-optimal match might not be assortative so as to preserve some information rents for buyers. Their setting matches each buyer with an exclusive seller, eliminating the inter-firm competition that is at the core of my analysis.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 analyzes the monopoly case. I first provide a simple condition under which consumer surplus can be increased by limiting consumers' access to the monopoly. Subsequently, I characterize the consumer-optimal access policy. Section 4 examines the oligopoly settings, demonstrating that, in most situations, the platform can increase consumer surplus by restricting their access to firms. Section 5 concludes the paper. All proofs are in the appendix.

[5] Condorelli and Szentes (2022b) studies the oligopoly setting but in Cournot quantity competition, which leads to very different forces.

2 Model

There are n firms in the market. Each produces a differentiated product at zero marginal cost.[6] Let $N := \{1, \ldots, n\}$ denote the set of all firms. There is one consumer in the market. The consumer has unit demand and has value v_i for firm i's product. I assume that each value v_i is in $[\underline{v}, 1]$, where I normalize the largest possible value to one and $\underline{v} \in [0, 1)$. The consumer's *type* is her value vector $v := (v_1, \ldots, v_n) \in \mathcal{V} := [\underline{v}, 1]^n$. If the consumer of type v purchases product i at a price of p_i, she obtains a payoff of $v_i - p_i$. If a consumer chooses not to purchase any product, then she obtains her outside option whose value I normalize to 0.

The consumer's type is drawn from the prior distribution $\mu \in \Delta(\mathcal{V})$ and is privately observed by the consumer.[7] I assume that prior distribution μ admits a density $f(\cdot)$ that has full support on $\mathcal{V}$ and $f(\cdot)$ is continuously differentiable on $\mathcal{V}$.

The platform observes each consumer's type and chooses her *consideration set*, namely the menu of products and price offers from which she chooses. To formalize how this consideration set is formed, let $\mathcal{P}(N)$ denote the set of all subsets of N. A matching ϕ is a mapping from the type space $\mathcal{V}$ to $\mathcal{P}(N)$:

$$\phi : \mathcal{V} \to \Delta(\mathcal{P}(N)).$$

This formulation allows the platform to randomly assign a set of firms to the consumer. I denote the set of all possible matchings by Φ. I distinguish between the matching in which every consumer accesses all firms from those in which some consumers do not.

Definition 1. *A matching ϕ is* ***full*** *if it matches each type to all firms: $\phi(N|v) = 1$ for all v in $\mathcal{V}$. By contrast, a matching ϕ is* ***partial*** *if it matches a positive measure of consumer types to a strict subset of firms, i.e., $\mu(\{v|\phi(N|v) < 1\}) > 0$.*

My analysis focuses on a simple kind of partial matching, namely the one that matches a consumer type to either all or no firms.

Definition 2. *A matching ϕ is* ***simple*** *if for all v in $\mathcal{V}$, $\phi(N|v) + \phi(\emptyset|v) = 1$.*

[6] I assume zero marginal cost for expositional convenience; the same analysis applies with heterogenous marginal cost.

[7] Another equivalent interpretation is the population interpretation: there is a continuum of consumers in the market and the distribution of consumer valuations is given by μ.

I denote the set of simple matching by Φ_o. For a simple matching ϕ, I simply use $\phi(v)$ to denote the probability that type v accesses all firms; with probability $1-\phi(v)$, type v is excluded and accesses no firm.

The platform chooses a matching ϕ. The matching induces the following game. First, each firm simultaneously sets a price for its product. Then, the platform matches these products and price offers to consumer types according to matching ϕ. The consumer then chooses which, if any, price offer to accept. I study the subgame perfect equilibria of this game.

For tractability, for each chosen matching $\phi \in \Phi$, I restrict my attention to pure strategy equilibria whenever they exist. I denote the set of pure strategy equilibrium by $\mathcal{E}(\phi)$. In the monopoly case, $\mathcal{E}(\phi)$ is the set of optimal monopoly prices. Given a matching ϕ and an equilibrium $p^* \in \mathcal{E}(\phi)$, consumer surplus is denoted by $CS(\phi, p^*)$:

$$CS(\phi, p^*) := \int_{v\in\mathcal{V}} \sum_{S\in\mathcal{P}(N)\setminus\{\emptyset\}} \max_{i\in S}\{(v_i - p_i^*)\mathbb{1}_{v_i\geq p_i^*}\}\phi(S|v)\mu(dv).$$

Definition 3. *Matching ϕ^* is **consumer-optimal** if there exists an equilibrium p^* under this matching such that the consumer surplus is higher than the consumer surplus of any equilibrium under any other matching:*

$$CS(\phi^*, p^*) \geq CS(\phi, \hat{p}) \text{ for all } \phi \in \Phi \text{ and } \hat{p} \in \mathcal{E}(\phi).$$

I say that a matching ϕ is **improvable** if it is not consumer-optimal. It turns out that for full matching to be improvable, the market cannot be fully covered in the sense that all types participate in trade:

Definition 4. *In the monopoly case, the market is **fully covered** if the lowest value $\underline{v}$ is an optimal price under full matching; in the oligopoly case, the market is **fully covered** if there exists an equilibrium under full matching that one firm's equilibrium price is weakly below the lowest value $\underline{v}$.*

In the monopoly case, if the market is fully covered, there is no room to further reduce the monopoly's price by restricting the consumer's access to the monopoly. In the oligopoly case, it is hard to improve consumer surplus in a fully covered market for a more subtle reason.

3 Monopoly

First, I study the setting with a single firm in the market. In this case, all matchings are simple matchings. I first obtain a simple condition under which full matching is improvable. Then I characterize the consumer-optimal matching.

Let $F(\cdot)$ denotes the CDF of μ and let $\pi(\cdot)$ denotes the profit function under full matching:

$$\pi(p) := p(1 - F(p)), p \in [0, 1].$$

Proposition 1. *Full matching is improvable when the market is* not *fully covered and the profit function under full matching, $\pi(\cdot)$, has a unique maximizer p^* and $\pi(\cdot)$ is strictly concave at this maximizer ($\pi''(p^*) < 0$).*

I offer a graphical sketch of the argument below; The formal proof is in the appendix. Note a unit-elastic demand curve (or equal revenue distribution) violates the conditions in Proposition 1 since the market is fully covered and under full matching, the monopoly is indifferent over a continuum of prices.

Sketch. In this sketch, I impose the stronger condition that $\pi(\cdot)$ is strictly concave with an interior maximizer. I first use Figure 3 to illustrate how to construct a matching that induces the monopoly to reduce its price by a small amount ε. Then I show when ε is sufficiently small, the price reduction gain offsets the loss from excluding consumers.

In panel (a) of Figure 3, the blue curve depicts $\pi(\cdot)$—the profit function under full matching. Since it is strictly concave, there is a unique maximizer p^*. If the monopoly reduces its price to $p^* - \varepsilon$, its profit will be reduced to $\pi(p^* - \varepsilon)$. Let A denotes this point—$A := (p^* - \varepsilon, \pi(p^* - \varepsilon))$. I draw the tangent of the blue curve at A. The tangent intersects the vertical axis at a profit level $\hat{\pi}$. Then I find the cutoff price above which profit is below $\hat{\pi}$ and denote this cutoff by c, as seen in panel (a) of Figure 3. I construct a matching $\hat{\phi}$ that excludes all types in the interval $[c, c']$ where c' is chosen so that the total measure of the excluded types equals the slope of the blue curve at A. Matching $\hat{\phi}$ is depicted by the blue curve in panel (b) of Figure 3.

Under matching $\hat{\phi}$, it is optimal for the monopoly to charge price $p^* - \varepsilon$: by charging this price, the monopoly obtains profit

$$\pi(p^* - \varepsilon) - (p^* - \varepsilon)\pi'(p^* - \varepsilon),$$

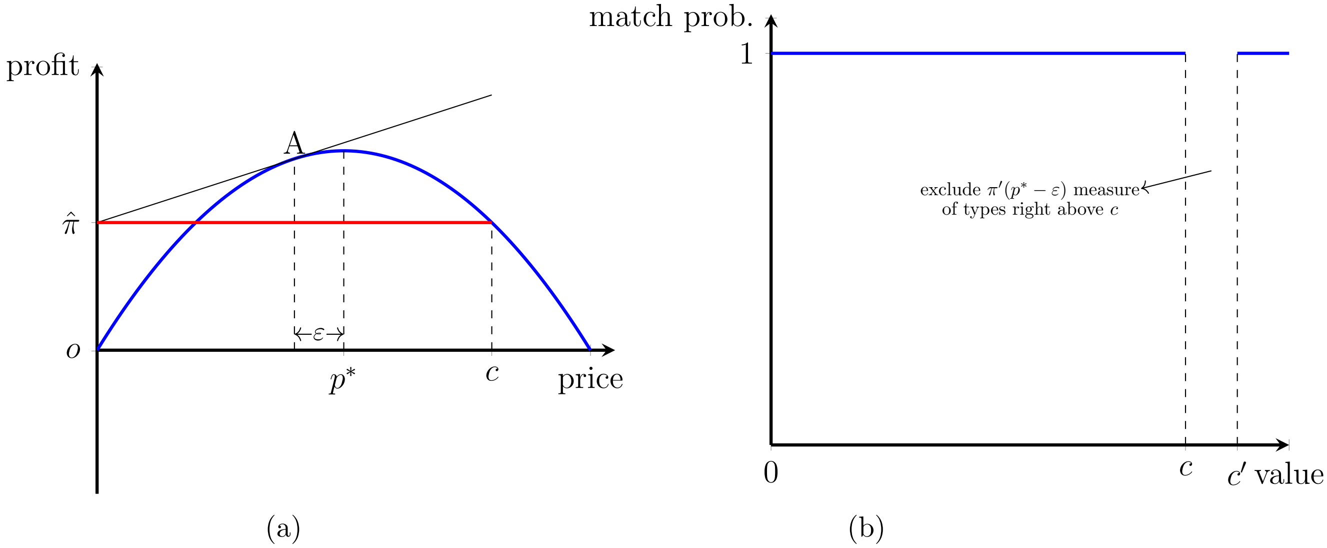


Figure 3: Construction of a matching that induces a small price reduction.

since $\pi'(p^* - \varepsilon)$ mass of consumers are excluded. Note, this profit exactly equals $\hat{\pi}$. Any price above c yields a profit below $\hat{\pi}$ even under full matching. Therefore, the monopoly has no incentive to deviate to prices above c. By charging price p below c, the monopoly obtains profit

$$\pi(p) - p\pi'(p^* - \varepsilon).$$

Again, this is because $\pi'(p^* - \varepsilon)$ mass of consumers are excluded. Since $\pi(\cdot)$ is concave, it is bounded by its linear approximation at point A:

$$\pi(p) \leq \pi(p^* - \varepsilon) - (p^* - \varepsilon - p)\pi'(p^* - \varepsilon).$$

This is illustrated by that in panel (a) of Figure 3, the blue curve is below the black tangent line through point A. Therefore, any price below c leads to a profit at most

$$\pi(p^* - \varepsilon) - (p^* - \varepsilon)\pi'(p^* - \varepsilon) = \hat{\pi}.$$

Hence, it is optimal for the monopoly to reduce its price to $p^* - \varepsilon$.

The resulting consumer surplus is strictly larger than under full matching if the price reduction gain offsets the loss from excluding consumers. A sufficient condition for this to happen is:

$$[1 - F(p^*) - \pi'(p^* - \varepsilon)]\,\varepsilon > \pi'(p^* - \varepsilon)(c' - p^*).$$

Since the price reduction gain is at least the left-hand side of this sufficient condition and the loss from excluding consumers is at most the right-hand side of this sufficient condition. Rearrange, this sufficient condition becomes

$$\frac{1 - F(p^*) - \pi'(p^* - \varepsilon)}{\pi'(p^* - \varepsilon)/\varepsilon} > (c' - p^*).$$

This sufficient condition holds for sufficiently small ε since it holds in the limit when ε goes to zero:

$$\frac{1 - F(p^*) - \pi'(p^* - \varepsilon)}{\pi'(p^* - \varepsilon)/\varepsilon} \to \frac{1 - F(p^*)}{-\pi''(p^*)} \text{ and } (c' - p^*) \to 0.$$

Note $\pi'(p^* - \varepsilon)$ converges to zero and c' converges to p^* as $\varepsilon \to 0$. Therefore, when ε is sufficiently small, matching $\hat{\phi}$ leads to a strictly larger consumer surplus than full matching.

□

Since full matching is not consumer-optimal, it raises the question of which matching is consumer-optimal. It turns out that the consumer-optimal matching induces a pointwise largest demand among matchings that lead to the same monopoly profit.

Let $D(\cdot|\phi)$ denote the demand function induced by each matching $\phi \in \Phi$:

$$D(p|\phi) := \int_p^1 f(v)\phi(v)dv, \ p \in [0,1]. \tag{1}$$

Definition 5. *A matching $\bar{\phi}$ is **extremal** if its induced demand function is pointwise larger than the demand function induced by any matching that leads to the same monopoly profit:*

$$D(\cdot|\bar{\phi}) \geq D(\cdot|\phi)\ \forall \phi \text{ that } \max_{p\in[0,1]} pD(p|\phi) = \max_{p\in[0,1]} pD(p|\bar{\phi}).$$

From Definition 5, it is not clear if extremal matchings exist and what form they may take. The main result of this paper constructs the extremal matching corresponding to each profit level and establishes that an extremal matching is consumer-optimal.

Let $\bar{\pi}$ denote the profit under full matching:

$$\bar{\pi} := \max_{p \in [0,1]} \pi(p).$$

Theorem 1. *There exists an extremal matching that is consumer-optimal.*

I offer a graphical sketch of the proof assuming that the demand function under full matching—$1 - F(\cdot)$ is concave; the formal proof (without this assumption) is in the appendix.

Sketch. Intuitively, no matching can lead to a profit above $\bar{\pi}$.

Given each profit level $\pi \in [0, \bar{\pi}]$, I construct the extremal matching that leads to profit π. It will follow from the construction that an extremal matching is consumer-optimal.

The extremal matching that leads to profit $\bar{\pi}$ is simply the full matching. The extremal matching that leads to profit 0 excludes almost all types of consumers with probability one.

Let Φ_π denote the set of matchings that lead to profit π:

$$\Phi_\pi := \{\phi \in \Phi | \max_{p \in [0,1]} pD(p|\phi) = \pi\}.$$

Fix $\pi \in (0, \bar{\pi})$. I illustrate the construction of the extremal matching in Φ_π by Figure 4.

Let ϕ be an arbitrary matching in Φ_π. I first construct a pointwise upper bound of $D(\cdot|\phi)$—the demand function induced by ϕ. Then I show that there exists an (almost unique) matching in Φ_π whose induced demand function equals this upper bound. It follows this matching is the extremal matching in Φ_π.

In Figure 4, I have prices on the vertical axis and demands on the horizontal axis. The blue curve depicts the demand function under full matching. Therefore, the graph of $D(\cdot|\phi)$ should be left of the blue curve. In addition, since $\phi \in \Phi_\pi$, I have

$$D(p|\phi) \leq \frac{\pi}{p}, \ \forall p > 0.$$

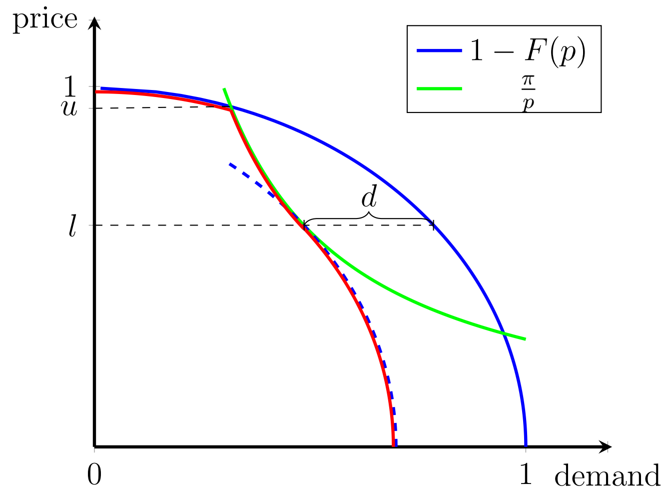


Figure 4: Construction of extremal matching in Φ_π

Therefore, the graph of $D(\cdot|\phi)$ should be left of the graph of $\frac{\pi}{p}$, which is depicted as the green curve.

Note at each price p, the horizontal gap between the graph of $D(\cdot|\phi)$ and the blue curve measures the total mass of consumer types excluded by matching ϕ that have values above p:

$$1 - F(p) - D(p|\phi) = \int_p^1 f(v)[1 - \phi(v)]dv.$$

Therefore, the horizontal gap between the graph of $D(\cdot|\phi)$ and the blue curve should be larger at lower prices. I argue above that the graph of $D(\cdot|\phi)$ should be left of the green curve. Note the green curve crosses the blue curve at price u, and the horizontal gap between the blue curve and the green curve keeps increasing as prices decrease from u to l and achieves maximum at price l. Denote this maximum gap by d. Therefore at each price below l, the horizontal distance of the blue curve and the graph of $D(\cdot|\phi)$ is at least d. Hence, the graph of $D(\cdot|\phi)$ should be left of the blue dashed curve, which is the horizontal shift of the blue curve to the left by a distance of d.

Therefore, a pointwise upper bound of $D(\cdot|\phi)$ is the red curve. Now I construct a matching that induces a demand function equal to the red curve. From Equation (1), this matching should be given by the slope of the red curve. Let D_π denote the function represented by the red curve (with the independent variable on the vertical axis and the dependent variable on the horizontal axis). Define

$$\bar{\phi}_\pi(v) := -\frac{D'_\pi(v)}{f(v)}, v \in [\underline{v}, 1],\ v \neq u.$$

I verify that $\bar{\phi}_\pi(\cdot)$ is a valid matching: since the red curve is downward sloping, $\bar{\phi}_\pi(\cdot) \geq 0$. In interval $(u, 1]$, the blue cure coincides with the red curve, therefore,

$$\bar{\phi}_\pi(v) = 1, v \in (u, 1].$$

In interval $[0, l)$, the red curve is a horizontal shift of the blue curve and, therefore has the same slope:

$$\bar{\phi}_\pi(v) = 1, v \in [\underline{v}, l].$$

At each point in the interval $(l, u]$, the blue curve is always flatter than the red curve, therefore

$$\bar{\phi}_\pi(v) < 1, v \in (l, u].$$

Therefore $\bar{\phi}_\pi(\cdot)$ is a valid matching and it only excludes intermediary types in interval $(l, u]$ with positive probability. By construction, matching $\bar{\phi}_\pi(\cdot)$ induces demand function D_π (the red curve). In addition, it follows from Figure 4 that $\bar{\phi}_\pi \in \Phi_\pi$. Hence, matching $\bar{\phi}_\pi(\cdot)$ is the extremal matching in Φ_π. Any other extremal matching in Φ_π must induce the same demand and therefore can only differ with matching $\bar{\phi}_\pi(\cdot)$ on a zero-measure set.

The extremal matching in Φ_π is, in fact, the consumer-optimal matching within set Φ_π: Note consumer surplus equals the area between the induced demand curve and the horizontal line at the optimal monopoly price. The extremal matching in Φ_π induces a demand function equal to the red curve, which is to the right of the graph of any demand function induced by any matching in set Φ_π. In addition, from Figure 4, the extremal matching in Φ_π induces the monopoly to reduce its price to l. No other matching in set Φ_π can induce a strictly lower optimal monopoly price than l: under any matching in set Φ_π, an optimal monopoly price p should generate a demand equal to $\frac{\pi}{p}$. That is a demand on the green curve. At the same time, the generated demand must be to the left of the red curve. Therefore, this optimal monopoly price has to be above l. Since among matching in set Φ_π, the extremal matching in Φ_π induces a pointwise largest demand and the lowest possible optimal monopoly price, the extremal matching in Φ_π is consumer-optimal within set Φ_π.

As a result, to find the consumer-optimal matching, I only need to optimize within the set of the extremal matching corresponding to each profit level π in $[0, \bar{\pi}]$.

$\square$

The above describes my main result, namely that limiting consumer choices can make them better off. Incidentally, my argument relies purely on price theory, and how excluding some consumers results in price discounts for the remainder.

A different objective is to maximize not consumer surplus but total surplus. Given a matching ϕ and an associated optimal price $\hat{p} \in \mathcal{E}(\phi)$, I denote total surplus by $T(\phi, \hat{p})$. Under this monopoly setting, the total surplus is calculated as :

$$T(\phi, \hat{p}) = \int_{\hat{p}}^{1} v f(v) \phi(v) dv.$$

Definition 6. *A matching ϕ^s is **social-optimal** if it maximizes total surplus among all matchings:*

$$\exists p^s \in \mathcal{E}(\phi^s), T(\phi^s, p^s) \geq T(\phi, p) \ \forall \phi \in \Phi \text{ and } p \in \mathcal{E}(\phi).$$

I show conditions under which the full matching is socially optimal; these conditions usually hold. But there are cases where even (utilitarian) efficiency is enhanced by removing types.

Proposition 2. *If besides conditions in Proposition 1, the density is downward sloping and has elasticity larger than one at the optimal price p^*—$\frac{d\log(f)}{d\log(p)}(p^*) < -1$, then full matching is* not *social-optimal.*

Proof. Note when conditions in Proposition 1 are satisfied, I can apply the same construction as in sketch proof of Proposition 1 to induce the monopoly to reduce its price by a sufficiently small amount ε.[8] A sufficient condition for this construction to result in a strictly larger total surplus is

$$\int_{p^*-\varepsilon}^{p^*} v f(v) dv > \pi'(p^* - \varepsilon) c'.$$

[8]Though, in the sketch, I assume $\pi(\cdot)$ is globally strictly concave; for sufficiently small ε, the same construction goes through when $\pi(\cdot)$-is locally concave—$\pi''(p^*) < 0$.

The left-hand side of this sufficient condition is the increase in total surplus due to extra consumers in the interval $[p^* - \varepsilon, p^*]$ participating in trade following the price reduction. The loss from excluding consumers is bounded by the right-hand side of this sufficient condition. Rearrange, this sufficient condition becomes

$$\frac{\int_{p^*-\varepsilon}^{p^*} vf(v)dv}{\pi'(p^*-\varepsilon)} - c' > 0.$$

If this condition holds in the limit as ε goes to zero, it will hold for sufficiently small ε. By L'Hospital's Rule,

$$\lim_{\varepsilon \to 0} \frac{\int_{p^*-\varepsilon}^{p^*} vf(v)dv}{\pi'(p^*-\varepsilon)} - c' = \frac{p^* f(p^*)}{-\pi''(p^*)} - p^*$$

Note $-\pi''(p^*) = 2f(p^*) + p^* f'(p^*)$. Therefore,

$$\frac{p^* f(p^*)}{-\pi''(p^*)} - p^* > 0 \iff \frac{d\log(f)}{d\log(p)}(p^*) < -1.$$

As a result, when conditions in Proposition 1 are satisfied and $\frac{d\log(f)}{d\log(p)}(p^*) < -1$, there exists sufficiently small ε such that the same construction as in sketch proof of Proposition 1 results in a striclty larger total surplus than full matching. $\square$

The Bounded Pareto distribution with support $[\underline{v}, 1]$ satisfies all conditions in Proposition 2 for certain parameter values. Recall the Bounded Pareto distribution with this support has density $f(v) = \frac{\alpha \underline{v}^\alpha}{(1-\underline{v}^\alpha)v^{\alpha+1}}$. The parameter α is the shape parameter and is required to be strictly positive and the minimum value $\underline{v}$ is bounded away from zero. It is straightforward to verify that when shape parameter α belongs to the interval $(0, 1)$ and the minimum value $\underline{v}$ is strictly below $(1-\alpha)^{\frac{1}{\alpha}}$, all conditions in Proposition 2 are satisfied, therefore full matching is not social-optimal.

When full matching is not social-optimal, there exists another extremal matching that is social-optimal.[9]

Theorem 2. *There always exists an extremal matching that is social-optimal. When density is log-concave, full matching is social-optimal.*

[9]Furthermore, under slightly stronger conditions, the set of extremal matching can achieve any *feasible* pair of consumer surplus and monopoly profit.

4 Oligopoly

In this section, I study the setting where multiple firms with differentiated products engage in Bertrand price competition. It is well-known in the literature that pure strategy equilibrium may not exist in these games. For tractability, I only consider pure strategy equilibrium in my analysis. Therefore, I assume that density is log-concave to ensure pure strategy equilibria exist under the full matching.

Theorem 3. *Full matching is improvable when the market is* not *fully covered and density is log-concave.*

The idea is to apply a similar analysis as in the proof of Proposition 1 since all the conditions there are satisfied when density is log-concave. Therefore, a similar logic goes through: the density being log-concave implies that each firm's demand given opponent firms' prices is log-concave in its own price. Therefore each firm's profit function is strictly log-concave in its own price and hence has a unique maximizer. Hence, under full matching, at the equilibrium with the highest consumer surplus, each firm's equilibrium price will be the unique best response to opponent firms charging equilibrium prices. In addition, I can find a small neighborhood of the equilibrium price vector so that within this neighborhood, each firm's profit is always strictly concave in its own price.

In this oligopoly setting, we obtain further insights about which consumers should be matched with which firms. Figure 5 shows a partial matching that has a larger consumer surplus than full matching. Let p^* denote the equilibrium with the largest consumer surplus under full matching. I depict the allocation under full matching and equilibrium p^* in panel (a). This equilibrium is not symmetric and firm 1 charges a lower price than firm 2 in equilibrium. Consumer types will choose the outside option if their value for each firm's product is lower than that firm's equilibrium price. Consumer types will purchase product i if their payoff from purchasing product i is positive and higher than the payoff from purchasing product j.

By excluding types (with probability one) in those gray regions in panel (b), the platform induces firm 1 to lower its price by ε and leads to a new equilibrium $(p_1^* - \varepsilon, p_2^*)$. Under full matching, given firm 2 charging price p_2^*, firm 1 does not want to charge price $p_1^* - \varepsilon$. Instead, firm 1 wants to increase its price to p_1^*. To discourage firm 1 from increasing its price, just like in the sketch proof of Proposition 1 above, the platform excludes a small square containing types that have values for product 1

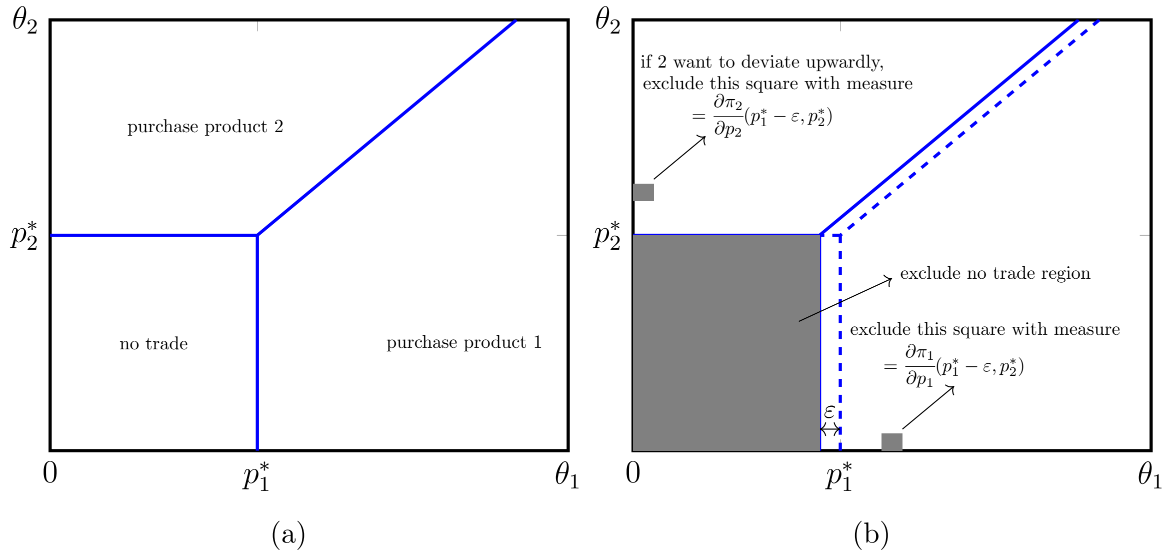


Figure 5: A Partial Matching with Larger Consumer Surplus than Full Matching

bounded away from p_1^* and values for product 2 very small. The size of the square is chosen so that the mass of types in this square equals to the slope of firm 1's profit function at the new equilibrium price $p_1^* - \varepsilon$ given firm 2 charging p_2^*.

Given that firm 1 sets price $p_1^* - \varepsilon$ and consumer types in the square are excluded, it is not clear if firm 2 wants to deviate from equilibrium price p_2^* and if it does, what directions firm 2 may deviate. This is because firms' pricing decisions in Bertrand's game with product differentiation do not satisfy strategic complementarity. To make sure that firm 2 doesn't lower prices, the platform further excludes consumer types in the no-trade region. This invokes no loss in consumer surplus since those types don't trade under the original equilibrium. If instead, firm 2 wants to increase prices (that is $\frac{\partial \pi_2}{\partial p_2}(p_1^* - \varepsilon, p_2^*) > 0$), then the platform further excludes a second square symmetric to the first square. This second square contains consumer types that have values for product 2 bounded away from p_2^* and values for product 1 very small. The size of the second square is chosen so that the mass of types in this square equals to the slope of firm 2's profit function at the equilibrium price p_2^* given firm 1 charging $p_1^* - \varepsilon$. Note by excluding the no-trade region and the second square, firm 1's incentive to deviate from the new equilibrium price is further reduced. Therefore, $(p_1^* - \varepsilon, p_2^*)$ is an equilibrium under this partial matching.

In certain scenarios, the platform cannot match consumers to no firm and can only exclude a small number of firms from a consumer's consideration set.

Definition 7. *A matching is* ***almost full*** *if it excludes at most one firm from each type of consumer's consideration set.*

Corollary 1. *Within almost full matchings, full matching is improvable under the same conditons in Theorem 3.*

In Figure 5, the almost full matching that improves upon full matching is as follows: for types in the two squares, the platform only excludes each type's preferred product from the type's consideration set. For consumers in the no-trade region, the platform only excludes firm 2 from their consideration set. In equilibrium, those types will choose the outside option and still not participate in trade. By mismatching in this way, the platform still induces the same equilibrium with firm 1 reducing its price by ε. The resulting consumer surplus is strictly larger.

I conclude this section by briefly discussing what matching the platform implements if the platform's objective is producer surplus (sum of firms' profits) instead of consumer surplus. In the monopoly setting, full matching always maximizes the monopoly's profit. In the oligopoly setting, full matching introduces competition between firms and hence cannot be producer optimal. The producer-optimal matching always matches each type of consumer exclusively to the most expensive product that the consumer is willing to purchase.

Theorem 4. *If a matching ϕ^p and an associated equilibrium p^* maximizes producer surplus across all matchings and equilibria, then under matching ϕ^p, each type of consumer is exclusively matched to the most expensive product that gives the consumer non-negative payoffs.*

5 Conclusion

Major internet platforms, such as Amazon and Google, are under heightened regulatory scrutiny from both U.S. and European authorities. Numerous investigations have revealed these platforms' practices of intentionally limiting consumers' access to relevant alternative choices. Contrary to the prevailing belief that limiting consumer choices invariably harms them, my research suggests that consumers can, in certain circumstances, benefit when platforms artificially restrict their choices.

In my model, consumers are fully rational and have complete knowledge about their preferences for various firms' products, without any attention or self-control costs. Firms offer differentiated products and engage in price competition. A platform chooses which firms appear in each consumer's consideration set. By strategically omitting some of the consumers' preferred choices, the platform steers the market towards an equilibrium where firms lower their prices, with some doing so strictly. Consequently, the consumer surplus in this equilibrium is strictly larger. This paper, therefore, presents a pricing-theoretic rationale for limiting consumer choice.

My research directly addresses a recent puzzling observation from an FTC investigation into Amazon. The FTC discovered that even when Amazon compromised the quality of its search results, it saw business growth rather than decline. My findings indicate that platforms, through strategic limitation of choice options, might indeed attract a larger user base.

The recent advancements in information design research shed light on the myriad ways consumers can benefit when engaging with internet platforms. For instance, Bergemann, Brooks, and Morris (2015) highlights the advantages stemming from market segmentation. Roesler and Szentes (2017) shows that consumers can benefit from choosing not to fully learn their own valuations. In a similar vein, Ali, Lewis, and Vasserman (2020) reveals the benefits consumers reap when they voluntarily disclose valuation information to firms. Adding to this discourse, my research offers a unique insight: unexpectedly, limiting consumer choice can boost consumer surplus.

In the monopoly setting, my research further elucidates the optimal design of restricting consumer choice. Under the consumer-optimal access policy, the platform determines the level of price reduction and the associated quantity of excluded consumers by striking an optimal trade-off between the advantages of lower prices and the loss from excluding consumers. The platform then carefully allocates the exclusion quotas among consumers of varied valuations, aiming to minimize the loss in consumer surplus while curbing the monopoly's price-increasing incentives. Ultimately, the platform chooses to exclude only those consumers of intermediate value. This results in a demand curve that locally mirrors a unit elastic demand curve, making the monopoly indifferent across a range of prices.

While my analysis emphasizes a single platform, it offers insights into platform competition. When platforms compete against each other and hence are driven to maximize consumer surplus, their rivalry doesn't always have to be zero-sum. By excluding certain consumers and letting them engage with competing platforms, a platform can indeed realize a greater consumer surplus for its users, making it more attractive to potential users.

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A Omitted Proofs

A.1 Proof of Proposition 1

Since the market is not fully covered, I have $p^* > \underline{v}$.

Given that $\pi''(p^*) < 0$, there exists a small interval $(p^* - \delta, p^* + \delta)$ such that $\pi''(p) < 0$ within this interval.

For a sufficiently small ε, I exclude consumers of measure ε with value right above $p^* + \delta$: I find a cutoff c above $p^* + \delta$ and excludes all consumer types in interval $(p^* + \delta, c)$. The cutoff c is chosen so that I exactly exclude measure ε of consumers.

Denote the new profit function by $\hat{\pi}(\cdot)$. I have

$$\hat{\pi}(p) = \pi(p) - p\varepsilon, \ \forall p \leq p^* + \delta.$$

For $p > p^* + \delta$, $\hat{\pi}(p) \leq \pi(p)$. When ε is sufficiently small, $\hat{\pi}(p^*)$ will be arbitrarily close to $\pi(p^*)$. In addition, since p^* is the unique maximizer, it follows:

$$\pi(p^*) > \pi(p), \ \forall p \notin (p^* - \delta, p^* + \delta).$$

Therefore, when ε is sufficiently small, the new optimal price remains within $(p^* - \delta, p^* + \delta)$. Note $\pi'(p)$ strictly decreases in the this interval and $\pi'(p^*) = 0$. Therefore, the new optimal price is unique and is given by the solution to the following equation.

$$\pi'(p) = \varepsilon.$$

Let this new optimal price be denoted by $\hat{p}$. It must be that $\hat{p} < p^*$. By the mean value theorem,

$$\pi'(\hat{p}) = \underbrace{\pi'(p^*)}_{=0} + \pi''(\xi)(\hat{p} - p^*), \tag{2}$$

where $\xi \in (\hat{p}, p^*)$.

When ε is sufficiently small, each excluded consumer type has a value arbitrarily close to $p^* + \delta$. This implies that the loss in consumer surplus from excluding these consumers is around $\varepsilon\delta$.

The benefit from the price reduction is at least:

$$(1 - F(p^*) - \varepsilon)(p^* - \hat{p}) = -\frac{\varepsilon}{\pi''(\xi)}(1 - F(p^*) - \varepsilon).$$

The equality is obtained by plugging in Equation (2). When both δ and ε are sufficiently small, the benefits from the price reduction will assuredly exceed the loss from excluding consumers.

A.2 Proof of Theorem 1

For each profit level $\pi \in (0, \bar{\pi}]$ and for each cutoff $c \in [0,1]$, define

$$G(c|\pi) := \max_{p\in[c,1]} \max\{1 - F(p) - \frac{\pi}{p}, 0\}.$$

Construct matching ϕ^*_π as follows:

$$\phi^*_\pi(v) := \begin{cases} \frac{\pi}{v^2 f(v)} & \text{if } G(\cdot|\pi) \text{ is differentiable at } v \text{ and } G'(v|\pi) < 0 \\ 1 & \text{otherwise} \end{cases}$$

I will show that matching ϕ^*_π is the π-extremal matching and there exists a $\pi^* \in (0, \bar{\pi}]$ such that matching $\phi^*_{\pi^*}$ is a consumer-optimal matching.

Fix a profit level $\pi \in (0, \bar{\pi}]$.

Step 1: I show that for each matching ϕ, if it induces monopoly profit π ($\phi \in \Phi_\pi$) then the measure of consumers excluded by matching ϕ with values above c is at least $G(c|\pi)$:

$$1 - F(c) - D(c|\phi) \geq G(c|\pi).$$

Since matching ϕ induces monopoly profit π, the induced demand is bounded from above by $\frac{\pi}{p}$

$$D(p|\phi) \leq \frac{\pi}{p},\ \forall p \in (0,1].$$

Consequently, the measure of consumers excluded by matching ϕ above each cutoff is bounded from below in a related way:

$$1 - F(p) - D(p|\phi) \geq 1 - F(p) - \frac{\pi}{p},\ \forall p \in (0,1]. \tag{3}$$

The left-hand side of Equation (3) is the cumulative measure of consumers excluded with a value above p:

$$1 - F(p) - D(p|\phi) = \int_p^1 f(\theta)(1 - \phi(\theta))d\theta.$$

Hence it is non-negative and decreasing in p. Therefore, for each cutoff $c \in [0, 1]$ and for each $p \geq c$,

$$1 - F(c) - D(c|\phi) \geq 1 - F(p) - D(p|\phi) \geq \max\{1 - F(p) - \frac{\pi}{p}, 0\},$$

which implies

$$1 - F(c) - D(c|\phi) \geq \max_{p \in [c,1]} \max\{1 - F(p) - \frac{\pi}{p}, 0\} = G(c|\pi).$$

Step 2: I show for each cutoff $c \in [0, 1]$, matching ϕ^*_π exactly excludes the least measure of consumers with values above c:

$$1 - F(c) - D(c|\phi^*_\pi) = G(c|\pi).$$

Note that function $G(c|\pi)$ is absolutely continuous in c. Hence, $G(c|\pi)$ has a derivative $G'(c|\pi)$ almost everywhere, the derivative is Lebesgue integrable, and

$$G(p|\pi) = \underbrace{G(1|\pi)}_{=0} - \int_p^1 G'(c|\pi)dc = \int_p^1 (-1)G'(c|\pi)dc, \ \forall p \in [0, 1].$$

By construction, $G(c|\pi)$ is decreasing in c, which implies $G'(c|\pi) \leq 0$. At each $c \in (0, 1)$ that $G'(c|\pi) < 0$, I have

(a): $G(c|\pi) = 1 - F(c) - \dfrac{\pi}{c}$;

(b): $G(c|\pi)$ and function $1 - F(c) - \dfrac{\pi}{c}$ has the same slope at c:

$$G'(c|\pi) = \frac{d\left(1 - F(c) - \frac{\pi}{c}\right)}{dc} = -f(c) + \frac{\pi}{c^2}.$$

Part (a) is true because otherwise, $G(p|\pi)$ will be a constant locally around $p = c$, violating $G'(c|\pi) < 0$. Part (b) is due to part (a) and that, by construction, $G(p|\pi)$ is pointwisely higher than $1 - F(p) - \frac{\pi}{p}$ for $p \in (0, 1]$.

Therefore,

$$(-1)G'(c|\pi) = f(c)[1 - \phi^*_\pi(c)] \text{ almost everywhere.}$$

Hence, for $p \in [0, 1]$,

$$G(p|\pi) = \int_p^1 (-1)G'(c|\pi)dc = \int_p^1 f(c)[1 - \phi^*_\pi(c)]dc = 1 - F(p) - D(p|\phi^*_\pi).$$

Note part (b) also implies $\phi^*_\pi(c) \leq 1$, hence ϕ^*_π is a valid matching.

Step 3: I show that matching ϕ^*_π is π-extremal and is consumer-optimal among matchings in Φ_π:

$$\phi^*_\pi \in \Phi_\pi, \text{ and } CS(\phi^*_\pi, \min \mathcal{E}(\phi^*_\pi)) \geq CS(\phi, \hat{p}), \ \forall \phi \in \Phi_\pi, \text{ and } \hat{p} \in \mathcal{E}(\phi).$$

For each matching $\phi \in \Phi_\pi$, combining step 1 and 2, I directly have that the demand induced by matching ϕ is bounded from above by the demand induced by matching ϕ^*_π—$D(\cdot|\phi) \leq D(\cdot|\phi^*_\pi)$. To show that the consumer surplus induced by ϕ is also bounded by the consumer surplus induced by ϕ^*_π—$CS(\phi, \hat{p}) \leq CS(\phi^*_\pi, \min \mathcal{E}(\phi^*_\pi))$, it is sufficient to show that matching ϕ cannot induce a monopoly price lower than the lowest price induced by matching ϕ^*_π—$\hat{p} \geq \min \mathcal{E}(\phi^*_\pi)$. This is because consumer surplus can be expressed as the area between the demand curve and the horizontal line crossing the monopoly price (price on the vertical axis and sales on the horizontal axis):

$$CS(\phi, \hat{p}) = \int_{\hat{p}}^1 vf(v)\phi(v)dv - \pi = \underbrace{\hat{p}D(\hat{p}|\phi) + \int_{\hat{p}}^1 D(v|\phi)dv}_{\text{integration by parts}} - \pi = \int_{\hat{p}}^1 D(v|\phi)dv. \quad (4)$$

The last equality is due to $\hat{p}D(\hat{p}|\phi) = \pi$ since ϕ induces monopoly profit π ($\phi \in \Phi_\pi$) and $\hat{p}$ is an optimal price under matching ϕ ($\hat{p} \in \mathcal{E}(\phi)$.

Note if matching ϕ^*_π induces a monopoly profit π ($\phi^*_\pi \in \Phi_\pi$), I immediately have the desired order of induced prices—$\hat{p} \geq \min \mathcal{E}(\phi^*_\pi)$: As argued right above, the profit at price $\hat{p}$ under matching ϕ exactly equals π—$\hat{p}D(\hat{p}|\phi) = \pi$. Suppose $\hat{p}$ is lower than the lowest price induced by matching ϕ^*_π—$\hat{p} < \min \mathcal{E}(\phi^*_\pi)$. Then $\hat{p}$ cannot be optimal price under matching ϕ^*_π—$\hat{p} \notin \mathcal{E}(\phi^*_\pi)$. If matching ϕ^*_π induces profit π—$\phi^*_\pi \in \Phi_\pi$, its induced profit under price $\hat{p}$ is strictly less than π—$\hat{p}D(\hat{p}|\phi^*_\pi) < \pi$. Therefore, at price $\hat{p}$, matching ϕ^*_π necessarily has a strictly lower demand than matching ϕ, contradicting that the demand induce by ϕ is bounded by the demand induced by ϕ^*_π—$D(\cdot|\phi) \leq D(\cdot|\phi^*_\pi)$.

To show matching ϕ^*_π induces profit π ($\phi^*_\pi \in \Phi_\pi$), I need to show: (a) the demand induced by ϕ is bounded by $\frac{\pi}{p}$; (b) there exists a price achieving profit π. Part (a) immediately follows from step 2:

$$1 - F(p) - D(p|\phi^*_\pi) = G(p|\pi) \geq 1 - F(p) - \frac{\pi}{p} \implies D(p|\phi^*_\pi) \leq \frac{\pi}{p}.$$

Define

$$p^*_\pi := \min \arg\max_{p \in (0,1]} 1 - F(p) - \frac{\pi}{p}.$$

Since $\pi \leq \bar{\pi}$, there exists a price p that leads to profit higher than π under full matching—$p(1 - F(p)) \geq \pi$, which implies $1 - F(p) - \frac{\pi}{p} \geq 0$. Therefore, $1 - F(p^*_\pi) - \frac{\pi}{p^*_\pi} \geq 0$. Hence, by construction of $G(\cdot|\pi)$ and p^*_π, I necessarily have $G(p^*_\pi|\pi) = 1 - F(p^*_\pi) - \frac{\pi}{p^*_\pi}$. Combining with step 2, I have $p^*_\pi D(p^*_\pi|\phi^*_\pi) = \pi$, therefore $\phi^*_\pi \in \Phi_\pi$.

In fact, p^*_π is exaclty the lowest monopoly price induced by matching ϕ^*_π—$\min \mathcal{E}(\phi^*_\pi) = p^*_\pi$: By construction of p^*_π, for each price p strictly lower than p^*_π,

$$1 - F(p) - \frac{\pi}{p} < 1 - F(p^*_\pi) - \frac{\pi}{p^*_\pi} = G(p^*_\pi|\pi),$$

combining with step 2, I have $pD(p|\phi^*_\pi) < \pi$, therefore $\min \mathcal{E}(\phi^*_\pi) = p^*_\pi$.

Hence, ϕ^*_π is an π-extremal matching, and any other π-extremal matching can only differ with ϕ^*_π on a zero measure set since they have to induce the same demand function.

Step 4: I show there exists a $\pi^* \in (0, \bar{\pi}]$ such that matching $\phi^*_{\pi^*}$ is a consumer-optimal matching.

Step 3 implies that to find consumer-optimal matching, it is without loss to restrict to matchings in set $\{\phi^*_\pi\}_{\pi \in (0,\bar{\pi}]}$.

Furthermore, I don't need to consider profit levels very close to 0 since a matching reducing the monopoly's profit close to zero necessarily excludes a majority of consumers, which leads to a very small consumer surplus.[10] Therefore, I can further restrict the set of target profit levels to $[\underline{\pi}, \bar{\pi}]$, where $\underline{\pi}$ is a sufficiently small positive number.

To show step 4, it is sufficient to show that the supremum of problem $\sup_{\pi \in [\underline{\pi}, \bar{\pi}], p \in \mathcal{E}(\phi^*_\pi)} CS(\phi^*_\pi, p)$ is attained. Therefore, I need to show

(a): Set $\{(\pi, p) | \pi \in [\underline{\pi}, \bar{\pi}], p \in \mathcal{E}(\phi^*_\pi)\}$ is close.

(b): Function $CS(\phi^*_\pi, p)$ is continuous in both π and p.

To show (a): Note by definition,

$$\mathcal{E}(\phi^*_\pi) = \arg\max_{p \in [0,1]} pD(p|\phi^*_\pi).$$

By step 2,

$$D(p|\phi^*_\pi) = 1 - F(p) - G(p|\pi).$$

Since $F(p)$ is continuous in p and note $G(p|\pi)$ is continuous both in p and π, I have $pD(p|\phi^*_\pi)$ is continuous both in p and π. Therefore, by the maximization theorem, $\mathcal{E}(\phi^*_\pi)$ is upper hemicontinuous in π with nonempty and compact values. Note the set in (a) is exactly the graph of $\mathcal{E}(\phi^*_\pi), \pi \in [\underline{\pi}, \bar{\pi}]$. Hence, by the closed graph theorem, this set is closed.

Part (b) follows from Equation (4) and that $D(p|\phi^*_\pi)$ is continuous both in p and π.

A.3 Proof of Theorem 2

Among matchings in Φ_π, the extremal matching is conditional social-optimal since it induces pointwise largest demand and lowest possible optimal monopoly prices. Therefore, there always exists an extremal matching that is social-optimal.

When density is log-concave, the demand under full matching $1 - F(\cdot)$ is log-concave. This is shown by Bagnoli and Bergstrom (2006).

[10] For any $k > 0$, if a matching reduces the monopoly profit to $\frac{1}{k^2}$, the measure of consumers matched with the monopoly with a value above $\frac{1}{k}$ is at most $\frac{1}{k}$. Therefore the induced consumer surplus is at most $\frac{2}{k}$.

By proof of Theorem 1, the conditional consumer optimal matching within set Φ_π is :

$$\phi_\pi^*(v) = \begin{cases} \frac{\pi}{v^2 f(v)} & \text{if } G(\cdot|\pi) \text{ is differentiable at } v \text{ and } G'(v|\pi^*) < 0 \\ 1 & \text{otherwise;} \end{cases}$$

For each $\pi \in (0, \bar{\pi})$, define

$$g(p|\pi) := 1 - F(p) - \frac{\pi}{p}, p \in (0, 1].$$

I show Lemma 1 in Appendix B:

Lemma 1. *When $1-F(\cdot)$ is log-concave, $\pi(\cdot)$ has a unique maximizer $p^* \in (0,1)$ and $\pi''(p^*) < 0$. In addition, for each $\pi \in (0, \bar{\pi})$, $g(p|\pi)$ has a unique maximizer $l(\pi) \in (0, p^*)$ and $g(p|\pi) = 0$ has a unique solution $u(\pi)$ in interval $(p^*, 1)$. When $p < l(\pi)$, $g(p|\pi)$ is strictly increasing; when $p \in (l(\pi), u(\pi))$, $g(p|\pi)$ is strictly decreasing; when $p > u(\pi)$, $g(p|\pi)$ is strictly negative.*

Therefore,

$$G(p|\pi) = \begin{cases} 0 & \text{if } p > u(\pi) \\ 1 - F(p) - \dfrac{\pi}{p} & \text{if } p \in [l(\pi), u(\pi)] \\ 1 - F(l(\pi)) - \dfrac{\pi}{l(\pi)} & \text{if } p < l(\pi). \end{cases}$$

Hence,

$$\phi_\pi^*(v) = \begin{cases} \frac{\pi}{v^2 f(v)} & \text{if } v \in [l(\pi), u(\pi)] \\ 1 & \text{otherwise;} \end{cases}$$

I show below full matching is social-optimal.

Fix a target profit level $\pi < \bar{\pi}$. The highest total surplus is achieved by matching ϕ_π^* and $l(\pi)$ as the selected price. The generated total surplus is

$$TS(\phi_\pi^*, l(\pi)) = \int_{l(\pi)}^1 v f(v) \phi_\pi^*(v) dv = \int_{l(\pi)}^{u(\pi)} \frac{\pi}{v} dv + \int_{u(\pi)}^1 v f(v) dv$$

$$= \pi[\log(u(\pi)) - \log(l(\pi))] + \int_{u(\pi)}^1 v f(v) dv.$$

$$\frac{dTS(\phi_\pi^*, l(\pi))}{d\pi} = \log(u(\pi)) - \log(l(\pi)) + \pi[\frac{u'(\pi)}{u(\pi)} - \frac{l'(\pi)}{l(\pi)}] - u(\pi)f(u(\pi))u'(\pi).$$

$$= \log(u(\pi)) - \log(l(\pi)) + [\frac{\pi}{u(\pi)} - u(\pi)f(u(\pi))]u'(\pi) - \pi\frac{l'(\pi)}{l(\pi)}.$$

Note,

$$[\frac{\pi}{u(\pi)} - u(\pi)f(u(\pi))]u'(\pi) = 1,$$

since, by the Implicit Function Theorem,

$$u'(\pi) = \frac{1}{1 - F(u(\pi)) - u(\pi)f(u(\pi))} = \frac{1}{\pi/u(\pi) - f(u(\pi))u(\pi)}.$$

By Implicit Function Theorem,

$$l'(\pi) = \frac{1}{2l(\pi)f(l(\pi)) + l^2(\pi)f'(l(\pi))}.$$

By proof of Lemma 1 part (e), the denominator is positive.

$$1 - \pi\frac{l'(\pi)}{l(\pi)} = 1 - \frac{l(\pi)f(l(\pi))}{2l(\pi)f(l(\pi)) + l^2(\pi)f'(l(\pi))} = \frac{f(l(\pi)) + l(\pi)f'(l(\pi))}{2f(l(\pi)) + l(\pi)f'(l(\pi))}.$$

From proof of Lemma 1,

$$f'(l(\pi)) \geq -\frac{f^2(l(\pi))}{1 - F(l(\pi))}.$$

Hence,

$$1 - \pi\frac{l'(\pi)}{l(\pi)} \geq \frac{f(l(\pi))[1 - \frac{l(\pi)f(l(\pi))}{1-F(l(\pi))}]}{2f(l(\pi)) + l(\pi)f'(l(\pi))} = \frac{\frac{f(l(\pi))}{1-F(l(\pi))}\pi'(l(\pi))}{2f(l(\pi)) + l(\pi)f'(l(\pi))} > 0.$$

Hence,

$$\frac{dTS(\phi_\pi^*, l(\pi))}{d\pi} = \log(u(\pi)) - \log(l(\pi)) + 1 - \pi\frac{l'(\pi)}{l(\pi)} > 0.$$

I show below that the set of extremal matching is sufficient to achieve all feasible surplus pairs when f is log-concave and $vf(v)$ is increasing.

Fix a target profit level $\pi \in [0, \bar{\pi}]$. Among matchings in Φ_π, the highest consumer surplus is achieved by matching ϕ_π^* with the lowest price $l(\pi)$ as the selected monopoly price. I only need to show that, under condition—$vf(v)$ increasing for $v \in [0,1]$, the lowest consumer surplus is achieved by matching ϕ^*_π with the highest price $u(\pi)$ as the selected monopoly price. The desired result will follow since $CS(\phi^*_\pi, p)$ is continuous function in p for $p \in [l(\pi), u(\pi)]$.

I first characterize the consumer-worst matching in Φ_π with a target price $\hat{p}$.

Lemma 2. *Fix a target price $\hat{p} \in [l(\pi), u(\pi)]$, the following matching minimize consumer surplus among matchings in Φ_π that lead the monopoly to charge $\hat{p}$:*

$$\hat{\phi}^*_\pi(v|\hat{p}) = \begin{cases} 1 & \textit{if } v \in [0, l(\pi)] \\ \frac{\pi}{v^2 f(v)} & \textit{if } v \in [l(\pi), \hat{p}] \\ 1 & \textit{if } v \in [\hat{p}, c(\pi,\hat{p})] \\ 0 & \textit{if } v \in [c(\pi,\hat{p}), 1] \end{cases}$$

, where $c(\pi,\hat{p})$ is a cutoff that solves

$$1 - F(c(\pi,\hat{p})) = 1 - F(\hat{p}) - \frac{\pi}{\hat{p}}.$$

Lemma 2 is proved in Appendix B.

Under matching $\hat{\phi}^*_\pi(\cdot|\hat{p})$, the generates consumer surplus is:

$$CS(\hat{p}) := \int_{\hat{p}}^{c(\pi,\hat{p})} vf(v)dv - \pi$$

and

$$F(c(\pi,\hat{p})) = F(\hat{p}) + \frac{\pi}{\hat{p}}$$

Hence,

$$CS'(\hat{p}) = c(\pi,\hat{p})f(c(\pi,\hat{p}))\frac{\partial c(\pi,\hat{p})}{\partial \hat{p}} - \hat{p}f(\hat{p}), f(c(\pi,\hat{p}))\frac{\partial(\pi,\hat{p})}{\partial \hat{p}} = f(\hat{p}) - \frac{\pi}{\hat{p}^2}$$

Hence,

$$CS'(\hat{p}) = c(\pi,\hat{p})[f(\hat{p}) - \frac{\pi}{\hat{p}^2}] - \hat{p}f(\hat{p}) = (c(\pi,\hat{p}) - \hat{p})f(\hat{p}) - \frac{c(\pi,\hat{p})\pi}{\hat{p}^2}.$$

By definition of $c(\pi, \hat{p})$,

$$F(c(\pi, \hat{p})) - F(\hat{p}) = f(\xi)(c(\pi, \hat{p}) - \hat{p}) = \frac{\pi}{\hat{p}}, \xi \in (\hat{p}, c(\pi, \hat{p}))$$

Hence,

$$CS'(\hat{p}) = \frac{\pi}{\hat{p}} \frac{f(\hat{p})}{f(\xi)} - \frac{c(\pi, \hat{p})\pi}{\hat{p}^2} = \frac{\pi}{\hat{p}} \left(\frac{f(\hat{p})}{f(\xi)} - \frac{c(\pi, \hat{p})}{\hat{p}} \right) < 0.$$

The inequality is due to $vf(v)$ increasing, hence,

$$\frac{f(\hat{p})}{f(\xi)} - \frac{c(\pi, \hat{p})}{\hat{p}} = \frac{\hat{p}f(\hat{p}) - f(\xi)c(\pi, \hat{p})}{f(\xi)\hat{p}} < \frac{\hat{p}f(\hat{p}) - f(\xi)\xi}{f(\xi)\hat{p}} \leq 0.$$

Hence, $CS(\hat{p})$ is minimized at $\hat{p} = u(\pi)$ and is exactly achieved by ϕ^*_π and $u(\pi)$ as selected price.

Remark: Note that for π close to π^*, even if $vf(v)$ is not increasing, $CS(\hat{p})$ still minimized at $\hat{p} = u(\pi)$ and ϕ^*_π. Because, $l(\pi), u(\pi) \to p^*, \pi \to \bar{\pi}$, Hence,

$$c(\pi, \hat{p}) \to 1, CS'(\hat{p}) \to (1 - p^*)f(p^*) - \frac{\pi^*}{(p^*)^2} = f(p^*)[1 - p^* - 1] < 0.$$

,since $\frac{\pi^*}{f(p^*)(p^*)^2} = 1$.

If $vf(v)$ is not increasing, for π small, there exists examples that $CS(\hat{p})$ is not minimized at $\hat{p} = u(\pi)$ and ϕ^*_π, and the set of extremal matching is not enough to achieves all feasible surplus pairs:

Example 1. *Let $f(v) = 2(1-v)$, note f is log-concave. $F(v) = 2v - v^2$. As $\hat{p} \to u(\pi)$,*

$$CS'(\hat{p}) \to (1-u(\pi))f(u(\pi)) - \frac{1 * \pi}{u(\pi)^2} = (1-u(\pi))f(u(\pi)) - \frac{1 - F(u(\pi))}{u(\pi)} = (1-u(\pi))^2[2-1/u(\pi)].$$

Note for π close to zero, $u(\pi) \to 1$, hence $CS'(p) > 0$. Therefore, matching $\hat{\phi}^*_\pi(\cdot|\hat{p})$ with $\hat{p} < u(\pi)$ achieves consumer surplus strictly lower than ϕ^*_π with $u(\pi)$ as selected price.

A.4 Proof of Theorem 3

Let $D_i(\cdot)$ denotes the demand of firm i under full matching:

$$\forall p \in [0,1]^n,\ D_i(p) := \Pr\left(\{v \in [0,1]^n | v_i \geq p_i, v_i - p_i \geq \max_{j \neq i}(v_j - p_j)\}\right).$$

Let $\pi_i(\cdot)$ denote the profit of firm i under full matching:

$$\pi_i(p) := p_i D_i(p_i, p_{-i}),\ \forall p \in [0,1]^n.$$

I show Lemma 3 in Appendix B.

Lemma 3. *When density is log-concave, the demand function of each firm i—$D_i(p_i, p_{-i})$ is log-concave in its own price p_i given any opponents' prices p_{-i}. In addition, the set of pure strategy equilibrium under full matching is nonempty and compact.*

Therefore, given any opponent firms' price p_{-i}, the profit function of each firm i—$\pi_i(p, p_{-i})$ has a unique maximizer. However, I cannot directly apply the logic of Proposition 1 because firstly, at a given equilibrium p^* under the full matching, $\pi_i(p_i, p^*_{-i})$ may not be second order differentiable in p_i at p^*. Secondly, even if the platform can exclude a small measure of consumers to induce one firm (say firm i) to reduce its price, it is not clear how other firms will best respond—they may increase or decrease their price.[11] In addition, I need to ensure pure strategy equilibrium still exists after I exclude some consumers. I adapt the proof of Proposition 1 to take care of these issues and show that the platform can exclude a small measure of consumers to induce a new equilibrium where one firm reduces its price slightly and it leads to a strictly larger consumer surplus.

By Lemma 3, the set of pure strategy equilibrium under full matching is nonempty and compact. Therefore, I can find an equilibrium p^* that has the highest consumer surplus within the set of equilibrium under full matching. Since the market is not fully covered, I have $p^*_i > \underline{v}$ for each i.

By Lemma 3, for each i, $\pi_i(\cdot, p^*_{-i})$ has a unique maximizer p^*_i. However, I don't have $\frac{\partial^2 \pi_i}{\partial p_i^2}(p^*) < 0$ since $\pi_i(p_i, p^*_{-i})$ generically is *not* second order differentiable in p_i

[11]Note firms' pricing decisions in Bertrand pricing game with differentiated products do not satisfy strategic complements. Here is an example: There are two firms and three types of consumers in the market—$(1,0), (0.5,0.5), (0,1)$, each with $\frac{1}{3}$ probability. As long as the opponent's price $p_{-i} > 0.5$, firm i will choose to slightly undercut its opponent and charge $p_{-i} - \varepsilon$. Instead, when $p_{-i} \leq 0.5$, firm i will forgo the middle type $(0.5, 0.5)$ and charges a price equal to 1.

at p^*. I resort to the following lemma, which also verifies the differentiability of $\pi_i(p)$ in p_i that will be used later.

Lemma 4. *For each i, profit function $\pi_i(p)$ is differentiable in p_i and marginal profit $\frac{\partial \pi_i}{\partial p_i}(p)$ is continuous in p. Profit function $\pi_i(p)$ is second order differentiable in p_i at each p that $p_i \neq p_j$ for any j. In addition, there exists a $\delta > 0$ and a corresponding neighborhood of p^*—$B(p^*, \delta) := \{p \in [0,1]^n : |p_i - p_i^*| \leq \delta, \ \forall i\}$ such that at each $p \in B(p^*, \delta)$ $\frac{\partial^2 \pi_i}{\partial p_i^2}(p) < 0$ whenever $\frac{\partial^2 \pi_i}{\partial p_i^2}(p)$ exists.*

I prove Lemma 4 in appendix B.

I relabel firms so that firm 1 has the lowest equilibrium price—$p_1^* \leq p_i^*, \ \forall i$. For a sufficiently small $\varepsilon < \delta$, let firm 1 reduces its price by ε and consider the new price vector $p(\varepsilon) := (p_1^* - \varepsilon, p_{-1}^*)$. Consider the following excluding matching—$\phi(\varepsilon)$: Firstly, matching $\phi(\varepsilon)$ excludes all consumers not purchasing any product under price $p(\varepsilon)$. In other words, it excludes all consumers in set $A(\varepsilon) = \{v | v_1 < p_1^* - \varepsilon \text{ and } v_i < p_i^*, \ \forall i \geq 2\}$. Second, for each firm i that has incentive to increase price under new price vector $p(\varepsilon)$ (that is firm i with $\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) > 0$), matching $\phi(\varepsilon)$ further excludes $\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))$ measure of consumers that has value for product i around $p_i^* + \delta$ and very small values for other products. In other words, matching $\phi(\varepsilon)$ further excludes consumers in neighborhood $C_i(\varepsilon) := \{v | v_i \in (p_i^* + \delta, p_i^* + \delta + \nu(\varepsilon)), v_j < \nu(\varepsilon), \ \forall j \neq i.\}$, where $\nu(\varepsilon)$ is chosen such that the measure of consumers in neighborhood $C_i(\varepsilon)$ is exactly $\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))$.

I claim when ε is sufficiently small, $p(\varepsilon)$ is an equilibrium under matching $\phi(\varepsilon)$: Given opponents' prices $p_{-i}(\varepsilon)$, firm i receives the following profit by setting price $p_i(\varepsilon)$:

$$\pi_i(p(\varepsilon)) - p_i(\varepsilon) \max\{\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)), 0\}.$$

Note as $\varepsilon \to 0$, this profit converges to $\pi_i(p^*)$ since $\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) \to \frac{\partial \pi_i}{\partial p_i}(p^*) = 0$. I first show that firm i will not deviate to prices outside interval $(p_i^* - \delta, p_i^* + \delta)$: Since $\pi_i(\cdot, p_{-i}^*)$ has a unique maximizer p_i^*,

$$\pi_i(p^*) > \max_{p_i \notin (p_i^* - \delta, p_i^* + \delta)} \pi_i(p_i, p_{-i}^*).$$

Note by deviating to a price p_i outside interval $(p_i^* - \delta, p_i^* + \delta)$, firm i's profit is at most $\pi_i(p_i, p_{-i}^*(\varepsilon))$, which converges to $\pi_i(p_i, p_{-i}^*)$ as $\varepsilon \to 0$. Hence when ε is sufficiently small, firm i will not deviate to price outside interval $(p_i^* - \delta, p_i^* + \delta)$.

Now consider possible deviations within interval $(p_i^* - \delta, p_i^* + \delta)$. Within this interval, $\pi_i(p_i, p_{-i}(\varepsilon))$ is strictly concave in p_i (because by Lemma 4, its first order derivatives is continuous with at most $N-1$ kinks and its second order derivatives (when exists) is strictly negative). I consider two cases.

Case 1—$\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) > 0$. In this case, when firm i deviates to a price $p_i \in (p_i^* - \delta, p_i^* + \delta)$, its profit is at most

$$\pi_i(p_i, p_{-i}(\varepsilon)) - p_i \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) \leq -p_i \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) + \underbrace{\pi_i(p(\varepsilon)) + \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))(p_i - p_i(\varepsilon))}_{\text{since } \pi_i \text{ is strict concave in } p_i.} = \underbrace{\pi_i(p(\varepsilon)) - p_i(\varepsilon) \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))}_{\text{profit if not deviate}}$$

Therefore, firm i has no strictly profitable deviation.

Case 2—$\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) \leq 0$ In this case, by not deviating and still charging price $p_i(\varepsilon)$, firm i obtains profit $\pi_i(p(\varepsilon))$. If firm i deviate to to a higher price $p_i \in (p_i(\varepsilon), p_i^* + \delta))$, its profit equals:

$$\pi_i(p_i, p_{-i}(\varepsilon)) \leq \pi_i(p(\varepsilon)) + \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))(p_i - p_i(\varepsilon)) \leq \pi_i(p(\varepsilon)).$$

The first inequality is again by strict concavity of $\pi_i(p_i, p_{-i}(\varepsilon))$ in p_i.

If firm i deviates to a lower price $p_i \in (p_i^* - \delta, p_i(\varepsilon))$, its profit

$$
\begin{aligned}
&= \pi_i(p_i, p_{-i}(\varepsilon)) - p_i \Pr(\{v \in A(\varepsilon) | v_i \geq p_i\}) \\
&\leq \pi_i(p(\varepsilon)) + \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))(p_i - p_i(\varepsilon)) - p_i \Pr(\{v \in A(\varepsilon) | v_i \geq p_i\}) \\
&\hspace{20em} (\pi_i \text{ is strict concave in } p_i) \\
&= \pi_i(p(\varepsilon)) + \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))(p_i - p_i(\varepsilon)) - p_i \int_{p_i}^{p_i(\varepsilon)} \int_{v_{-i} \leq p_{-i}(\varepsilon)} f(v_i, v_{-i}) dv_{-i} dv_i \\
&\hspace{20em} (\text{By definition of } A(\varepsilon)) \\
&\leq \pi_i(p(\varepsilon)) + \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon))(p_i - p_i(\varepsilon)) - p_i \min_{x \in [p_i^* - \delta, p_i^* + \delta]} \int_{v_{-i} \leq p_{-i}(\varepsilon)} f(x, v_{-i}) dv_{-i} (p_i(\varepsilon) - p_i) \\
&\leq \pi_i(p(\varepsilon)) + \left(- \frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) - (p_i^* - \delta) \min_{x \in [p_i^* - \delta, p_i^* + \delta]} \int_{v_{-i} \leq p_{-i}(\varepsilon)} f(x, v_{-i}) dv_{-i} \right) (p_i(\varepsilon) - p_i) \\
&\hspace{20em} (p_i > p_i^* - \delta) \\
&\leq \pi_i(p(\varepsilon)) \text{ for sufficiently small } \varepsilon \hspace{8em} \left(\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) \to 0, \varepsilon \to 0\right)
\end{aligned}
$$

Hence, firm i does not have strictly profitable deviations. Therefore, I have shown that $p(\varepsilon)$ is an equilibrium under matching $\phi(\varepsilon)$.

Now I compare the price reduction gain and loss from excluding consumers when moving from full matching and equilibrium p^* to partial matching $\phi(\varepsilon)$ and new equilibrium $p(\varepsilon)$. Note consumers that purchase product 1 under full matching and equilibrium p^* will continue to purchase product 1 under matching $\phi(\varepsilon)$ (once matched) and equilibrium $p(\varepsilon)$. Therefore, the price reduction gain is at least:

$$
\varepsilon \left(\underbrace{\Pr(\{v | v_1 \geq p_1^* \text{ and } v_1 - p_1^* \geq \max_{i>1} \{v_i - p_i^*\}\})}_{\text{Mass of consumers buy good 1 under } p^* \text{ and full matching}} - \max\{\frac{\partial \pi_1}{\partial p_1}(p(\varepsilon)), 0\} \right)
$$

While the loss of consumer surplus from excluding consumers is around,

$$
\sum_i \max\{\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)), 0\} \delta
$$

I show the following lemma in the appendix B:

Lemma 5. *Given $p_{-1}(\varepsilon)$, $\frac{\partial \pi_i}{\partial p_i}(p_1, p_{-1}(\varepsilon))$ is continuously differentiable in p_1 over interval $[p_1^* - \varepsilon, p_1^*]$.*

Note $p_{-1}(\varepsilon) = p_{-1}^*$. Therefore, by the mean value theorem,

$$\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) = \underbrace{\frac{\partial \pi_i}{\partial p_i}(p^*)}_{=0} - \frac{\partial^2 \pi_i}{\partial p_i \partial p_1}(\hat{p}_1, p_{-1}^*)\varepsilon,$$

where $\hat{p}_1 \in (p_1^* - \varepsilon, p_1^*)$. Hence,

$$\frac{\partial \pi_i}{\partial p_i}(p(\varepsilon)) = -\left(\text{left hand derivative of } \frac{\partial \pi_i}{\partial p_i}(p_1, p_{-1}^*) \text{ at } p_1 = p_1^* \right) \varepsilon + o(\varepsilon).$$

Therefore, when ε and δ are sufficiently small, matching $\phi(\varepsilon)$ and equilibrium $p(\varepsilon)$ has a strictly larger consumer surplus than full matching and equilibrium p^*.

A.5 Proof of Theorem 4

Let I denote the set of permutations of $(1, \ldots, n)$. In other words, I is the set of all bijections from N to itself. Let $i = (i_1, \cdots, i_n)$ denote a generic element of I. Each $i \in I$ is a rank of the n firms.

I will show that the producer optimal matching is characterized by optimal ranking i^* and optimal cutoff c^*, and matches consumers to firms sequentially according to ranking i^*: If $v_{i_1^*} \geq c_1^*$, the consumer is only matched to firm i_1^*; $\cdots$; if $v_{i_j^*} < c_j^*$ for each $j < k$ but $v_{i_k^*} \geq c_k^*$, the consumer is only matched to firm i_k^*; $\cdots$; if $v_{i_j^*} < c_j^*$ for each $j < n$, the consumer is only matched to firm i_n^*. Under this matching, firm i_k^* sets its price at c_k^* in equilibrium.

The optimal ranking i^* and the optimal cutoff c^* is given by the following two-step optimization problem:

$$\max_{i \in I} \max_{1 \geq c_1 \geq \cdots \geq c_n \geq 0} \sum_{k=1}^{k=n} c_k \int_{v_k \geq c_k, v_i < c_i \forall i < k} f(v) dv.$$

The following proof demonstrates the intuition behind Theorem 4: to maximize producer surplus, the platform creates monopoly positions for firms and matches each consumer to the most expensive product that the consumer is willing to purchase. Consider any matching $\phi \in \Phi$ and any pure strategy equilibrium $p \in \mathcal{E}(\phi)$. Rank firms by the price of their products in decreasing order and denote the rank by i. Hence, $p_{i_1} \geq, \cdots, p_{i_n}$. Consider the following new matching: If $v_{i_1} \geq p_{i_1}$, the consumer is only matched to firm i_1; $\cdots$; if $v_{i_j} < p_{i_j}$ for each $j < k$ but $v_{i_k} \geq p_{i_k}$, the consumer is only matched to firm i_k; $\cdots$; if $v_{i_j} < p_{i_j}$ for each $j < n$, the consumer is only matched to firm i_n.

Note, under this new matching, each firm is a local monopoly, and each consumer is matched to the most expensive product among those with a non-negative net utility to the consumer. Hence, the new matching has a weakly larger producer surplus if each firm i_k sets its price at p_{i_k}. The generated producer surplus is exactly:

$$\sum_{k=1}^{k=n} p_{i_k} \int_{v_{i_k} \geq p_{i_k}, v_{i_j} < p_{i_j} \forall j < k} f(v)dv.$$

Hence,

$$\max_{\phi \in \Phi} \max_{p \in \mathcal{E}(\phi)} \sum_{i \in N} \pi_i(\phi, p) \leq \max_{i \in I} \max_{1 \geq c_1 \geq \cdots \geq c_n \geq 0} \sum_{k=1}^{k=n} c_k \int_{v_k \geq c_k, v_i < c_i \forall i < k} f(v)dv.$$

I finish the proof by showing that this inequality actually holds with equality—under the above matching, it is optimal for each firm i_k^* to set price at c_k^*. Suppose not—under the above matching, there exists a firm i_k^* and a price $p' \neq c_k^*$ that gives firm i_k^* a profit strictly higher than the profit at price c_k^*. Let firm i_k^* change its price to p'. The generated producer surplus will be higher than the value of the two-step optimization problem. Given the new price vector after firm i_k^* changes its price to p', I can generate a new matching based on the above procedure, which further increases the producer surplus. Hence, I find a new (i', c') that gives the objective in the two-step optimization problem a higher value, which violates the optimality of (i^*, c^*).

The producer-optimal matching has two notable features. First, it intrinsically treats firms unequally—firms have different priorities in the matching. Second, it mismatches consumers to firms—some consumers are not matched to their favorite product. Note these are true even for symmetric value distributions. Under a symmetric value distribution, the optimal rank i^* in the producer-optimal matching is indeterminate. Because, given any matching and pure strategy equilibrium, any permutation of the firms' identities generates an equivalent matching and equilibrium with the same consumer surplus and producer surplus.

B Proofs of Technical Lemmas

Note in Appendix B, I use θ to denote value vector instead of v.

B.1 Proof of Lemma 1

I first show when $1 - F(\cdot)$ is log-concave,

(a): $\frac{f(p)}{1-F(p)}$ is increasing in interval $[0, 1)$.

(b): $\pi(\cdot)$ has a unique maximizer p^*. In addition, $\pi'(p) > 0$ when $p < p^*$ and $\pi'(p) < 0$ when $p > p^*$.

(c): $\pi''(p^*) < 0$.

(d): for $p \in (0, 1)$

$$f'(p) \geq -\frac{f^2(p)}{1 - F(p)}$$

(e): $p^2 f(p)$ strictly increase in interval $[0, p^*]$ and $\pi(p^*) = (p^*)^2 f(p^*)$

For (a): when $1-F(\cdot)$ is log-concave, $\log(1-F(p)$ is concave. Therefore $\frac{d\log(1-F(p))}{dp} = \frac{-f(p)}{1-F(p)}$ is decreasing.

For (b): Note $\pi(p) = p(1-F(p))$. When $1-F(\cdot)$ is log-concave, $\log(\pi(\cdot))$ is strictly concave. Therefore, $\log(\pi(\cdot))$ has a unique maximizer $p^* \in (0, 1)$ and $\frac{d\log(\pi(p))}{dp} = \frac{\pi'(p)}{\pi(p)}$ is strictly decreasing and equals zero at $p = p^*$. Hence, p^* is also the unique maximizer of $\pi(\cdot)$ and $\pi'(p) > 0$ when $p < p^*$ and $\pi'(p) < 0$ when $p > p^*$.

For (c): Note for $p \in (0, 1)$,

$$\pi'(p) = 1 - F(p) - pf(p) = (1 - F(p)) \left(1 - p\frac{f(p)}{1 - F(p)}\right).$$

Therefore,

$$\pi''(p^*) = -f(p^*) \underbrace{\left(1 - p^* \frac{f(p^*)}{1 - F(p^*)}\right)}_{=0 \text{ since } \pi'(p^*)=0} + (1 - F(p^*)) \frac{d\left(1 - p\frac{f(p)}{1-F(p)}\right)}{dp}(p^*) < 0$$

The last inequality is by part (a).

For part (d): When $1 - F(\cdot)$ is log-concave, $\frac{d^2 \log(1-F(p))}{dp^2} \leq 0$.

$$\frac{d \log(1 - F(p))}{dp} = \frac{-f(p)}{1 - F(p)}$$

$$\frac{d^2 \log(1 - F(p))}{dp^2} = \frac{-f'(p)(1 - F(p)) - f^2(p)}{(1 - F(p))^2} \leq 0$$

$$\implies f'(p) \geq -\frac{f^2(p)}{1 - F(p)}$$

For part (e):

$$\frac{dp^2 f(p)}{dp} = 2pf(p) + p^2 f'(p) \geq 2pf(p) - \frac{p^2 f^2(p)}{1 - F(p)} = pf(p)\left(2 - \frac{pf(p)}{1 - F(p)}\right).$$

The inequality is by part (d). Note for $p \in (0, p^*)$,

$$\pi'(p) > 0 \implies 1 - F(p) - pf(p) > 0 \implies \frac{pf(p)}{1 - F(p)} < 1.$$

Hence for each $p \in (0, p^*)$,

$$\frac{dp^2 f(p)}{dp} > 0.$$

To see $(p^*)^2 f(p^*) = \pi(p^*)$:

$$\pi'(p^*) = 0 \implies 1 - F(p^*) = p^* f(p^*) \implies (p^*)^2 f(p^*) = p^*(1 - F(p^*)) = \pi(p^*).$$

To finish proving Lemma 1, I need to find $u(\pi)$ and $l(\pi)$ so that the claims about $g(\cdot|\pi)$ hold. Note $\bar{\pi} = \pi(p^*)$. For each $\pi \in (0, \bar{\pi})$, by part (b), I define $u(\pi)$ to the unique solution of $\pi(p) = \pi$ in interval $(p^*, 1)$. By part (e), I define $l(\pi)$ to be the unique solution of $p^2 f(p) = \pi$ in interval $(0, p^*)$. By definition of $u(\pi)$ and part (b), I have $\pi(u(\pi)) = \pi$ and $\pi(p) < \pi$, $\forall p > u(\pi)$. Since $\pi(p) = p(1 - F(p))$, I have $g(u(\pi)) = 0$ and $g(p) < 0$ for $p > u(\pi)$. Note $g'(p) = -f(p) + \frac{\pi}{p^2}$. By part (e), I have $g'(p) > 0$ for $p \in (0, l(\pi))$ and $g'(p) < 0$ for $p \in (l(\pi), p^*)$. To show $g'(p) < 0$ in interval $(p^*, u(\pi))$: Note by part (b), in this interval, $\pi'(p) < 0$ and $\pi(p) > \pi$.

Therefore,

$$\pi'(p) = 1 - F(p) - pf(p) < 0 \implies \pi < \pi(p) = p(1 - F(p)) < p^2 f(p) \implies g'(p) < 0.$$

Therefore, $l(\pi)$ is the unique maximizer of $g(\cdot)$, $g(p)$ strictly increase in interval $[0, l(\pi)]$ and strictly decreases in interval $(l(\pi), u(\pi)$ and $g(p) < 0$ for $p > u(\pi)$. This finishes the proof of Lemma 1.

B.2 Proof of Lemma 2

Note we target profit level π and price $\hat{p}$. To achieve this target, the total mass of consumers with a value above $\hat{p}$ that need to be excluded equals

$$1 - F(\hat{p}) - \frac{\pi}{\hat{p}}.$$

This quantity is non-negative since $\hat{p} \in [l(\pi), u(\pi)]$, which implies $\pi(\hat{p}) \geq \pi$. In fact this quantity is strictly positive if $\hat{p} < u(\pi)$.

The above matching $\hat{\phi}^*_\pi(\cdot|\hat{p})$ excludes this mass of consumers that have the highest possible values. Hence, it has the lowest consumer surplus. To finish the proof, I only need to verify that $\hat{\phi}^*_\pi(\cdot|\hat{p}) \in \Phi_\pi$ and $\hat{p} \in \mathcal{E}(\hat{\phi}^*_\pi(\cdot|\hat{p}))$. Note for each $p \leq \hat{p}$,

$$D(p|\hat{\phi}^*_\pi(\cdot|\hat{p})) = \int_p^{\hat{p}} f(v)\hat{\phi}^*_\pi(v|\hat{p})dv + D(\hat{p}|\hat{\phi}^*_\pi(\cdot|\hat{p})) = D(p|\phi^*_\pi).$$

The last equality is because ϕ^*_π and $\hat{\phi}^*_\pi(\cdot|\hat{p})$ coincide in interval $[0, \hat{p}]$ and $\hat{p}D(\hat{p}|\phi^*_\pi) = \pi$.

For $p \in (\hat{p}, u(\pi)]$, $D(p|\hat{\phi}^*_\pi(\cdot|\hat{p})) \leq D(p|\phi^*_\pi)$ if $p \geq c(\pi, \hat{p})$. If $p < c(\pi, \hat{p})$,

$$D(p|\hat{\phi}^*_\pi(\cdot|\hat{p})) = 1 - F(p) - (1 - F(c(\pi, \hat{p}))) = \frac{\pi}{\hat{p}} - (F(p) - F(\hat{p})) \leq \frac{\pi}{\hat{p}} - \int_{\hat{p}}^p f(v)\phi^\pi(v)dv = D(p|\phi^*_\pi).$$

For $p > u(\pi)$,

$$D(p|\hat{\phi}^*_\pi(\cdot|\hat{p})) \leq 1 - F(p) = D(p|\phi^*_\pi).$$

Hence, I verified that $\hat{\phi}^*_\pi(\cdot|\hat{p}) \in \Phi_\pi$ and $\hat{p} \in \mathcal{E}(\hat{\phi}^*_\pi(\cdot|\hat{p}))$.

B.3 Proof of Lemma 3

This proof uses theorem 1 in Prékopa (1973). Let P be a probability measure generated by density f: for any measurable set C, $P(C) := \int_{\theta \in C} f(\theta) d\theta$. This theorem states that if f is log-concave, then

$$P(\lambda A + (1-\lambda)B) \geq P^{\lambda}(A) P^{1-\lambda}(B).$$

for any convex set A, B and any $\lambda \in (0,1)$. Here, sign $+$ means the Minkowski addition of sets:

$$\lambda A + (1-\lambda)B = \{\lambda\theta + (1-\lambda)\theta' | \theta \in A, \theta' \in B\}.$$

Let $N' = \{0\} \cup N$, $\theta_0 = p_0 = 0$. For each firm i, given opponents' prices p_{-i}, type θ will have the following reservation price for product i:

$$r_i(\theta | p_{-i}) := \min_{j \in N', j \neq i} \{\theta_i - (\theta_j - p_j)\}.$$

When firm i charges price $p \in [0,1]$, type θ will purchase product i if and only if $r_i(\theta_i | p_{-i}) \geq p$. Therefore,

$$D_i(p, p_{-i}) = \Pr(\{\theta | r_i(\theta_i | p_{-i}) \geq p\}).$$

Note $\{\theta | r_i(\theta | p_{-i}) \geq p\}$ is convex set. To show that $D_i(p, p_{-i})$ is log-concave in p, it is sufficient to show, for any $p, p', \lambda \in [0,1]$ and $\bar{p} = \lambda p + (1-\lambda)p'$,

$$\lambda\{\theta | r_i(\theta | p_{-i}) \geq p\} + (1-\lambda)\{\theta | r_i(\theta | p_{-i}) \geq p'\} \subseteq \{\theta | r_i(\theta | p_{-i}) \geq \bar{p}\},$$

which is straightforwardly true.

Given opponent prices p_{-i}, let $b_i(p_{-i})$ denote the best response of firm i,

$$b_i(p_{-i}) := \arg\max_{q \in [0,1]} q D_i(q, p_{-i}).$$

Since $D_i(q, p_{-i})$ is log-concave in q, the solution to this profit maximization problem of firm i is unique. Hence, the best response $b_i(\cdot)$ is a function, and by maximization theorem, it is also continuous in p_{-i}. Let $b : [0,1]^n \to [0,1]^n$ denotes the best response mapping, where $b_i(p) := b_i(p_{-i})$. The best response mapping is a continuous mapping from $[0,1]^n$ to itself, hence by Brouwer fixed-point theorem, the best response mapping has at least one fixed point. By definition, each fixed point of the best response mapping is a pure strategy equilibrium under full matching. Hence, the set of pure strategy equilibrium under full matching is non-empty. The set of pure strategy equilibrium is a subset of $[0,1]^n$. This set is close because mapping $b(p) - p$ is also continuous, hence the inverse image of the singleton set $\{0\}$ is close.

B.4 Proof of Lemma 4

step 1: I show

- $\pi_i(p)$ is differentiable in p_i;
- $\frac{\partial \pi_i}{\partial p_i}(p)$ is continuous in p
- $\pi_i(p)$ is second order differentiable in p_i at each p that $p_i \neq p_j$ for any j

Note $\pi_i(p) = p_i D_i(p_i, p_{-i})$. I only need to show the same properties holds for D_i.

Given a price vector $p \in (0,1)^n$, let $A_0^i(p)$ denote the event that consumers choose to purchase product i and the best alternative option is to not purchase any products:

$$A_0^i(p) := \{\theta | \theta_i \geq p_i, \text{ and } \theta_j < p_j, \ \forall j \neq i\}.$$

For each $j \neq i$, let $A_j^i(p)$ denote the event that consumers choose to purchase product i and the best alternative option is to purchase product j:

$$A_j^i(p) := \{\theta | \theta_i - p_i \geq \theta_j - p_j, \text{ and } \theta_j \geq p_j, \theta_j - p_j \geq \max_{k \in N \setminus \{i,j\}} (\theta_k - p_k)\}.$$

Note,

$$D_i(p) = \Pr\left(\cup_{j \in N \cup \{0\}, j \neq i} A_j^i(p)\right).$$

The intersection of any two sets in $\{A_j^i(p)\}_{j \in N \cup \{0\}, j \neq i}$ has zero measure, therefore

$$D_i(p) = \sum_{j \in N \cup \{0\}, j \neq i} \underbrace{\Pr(A_j^i(p))}_{:= \Psi_j^i(p)}.$$

I only need to show that Ψ_j^i has the same properties for each $j \in N \cup \{0\}, j \neq i$:

- $\Psi_j^i(p)$ is differentiable in p_i;
- $\frac{\partial \Psi_j^i}{\partial p_i}(p)$ is continuous in p;
- $\Psi_j^i(p)$ is second order differentiable in p_i at each p that $p_i \neq p_j$.

$$\Psi_0^i(p) = \int_{p_i}^1 \int_{\theta_{-i} \leq p_{-i}} f(\theta_i, \theta_{-i}) d\theta_{-i} d\theta_i$$

Here, for two vectors a, b, $a \leq b$ means $a_i \leq b_i$ for each coordinate i. Note Ψ_0^i is differentiable in p_i and

$$\frac{\partial \Psi_0^i}{\partial p_i}(p) = -\int_{\theta_{-i} \leq p_{-i}} f(p_i, \theta_{-i}) d\theta_{-i},$$

which is continuous in p. Ψ_0^i is second order differentiable in p_i:

$$\frac{\partial^2 \Psi_0^i}{\partial p_i^2}(p) = -\int_{\theta_{-i} \leq p_{-i}} \frac{\partial f}{\partial \theta_i}(p_i, \theta_{-i}) d\theta_{-i},$$

which is still continuous in p since I have assumed that f is continuously differentiable.

For $j \in N, j \neq i$,

$$\Psi_j^i(p) = \int_{p_j}^{\min\{1+p_j-p_i, 1\}} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} \int_{\theta_j - p_j + p_i}^1 f(\theta_i, \theta_j, \theta_{-(i,j)}) d\theta_i d\theta_{-(i,j)} d\theta_j$$

Here, for a vector a and a scalar c, $a \leq c$ means $a_i \leq c$ for each coordinate i. Note if $\theta_j > 1 + p_j - p_i$, then even if $\theta_i = 1$, type θ will not purchase product i. Therefore, I only integrate θ_j till $\min\{1 + p_j - p_i, 1\}$. Note Ψ_j^i potentially has a kink at p with $p_i = p_j$ (but is continuous at this point). We will see later that this point is not a real kink. First, Ψ_j^i is differentiable in p_i at each p with $p_i \neq p_j$:

$$\frac{\partial \Psi_j^i}{\partial p_i}(p) = \begin{cases} \int_{p_j}^1 \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} (-1) f(\theta_j - p_j + p_i, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j & \text{if } p_i < p_j \\ \int_{p_j}^{1+p_j-p_i} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} (-1) f(\theta_j - p_j + p_i, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j & \text{if } p_i > p_j \end{cases}$$

At p with $p_i = p_j$,

$$\frac{\partial_+ \Psi^i_j}{\partial p_i}(p) = \frac{\partial_- \Psi^i_j}{\partial p_i}(p) = \int_{p_j}^{1} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} (-1) f(\theta_j, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j$$

Hence, Ψ^i_j does not have kink at p with $p_i = p_j$ and is differentiable in p_i at each $p \in (0,1)^n$, and

$$\frac{\partial \Psi^i_j}{\partial p_i}(p) = -\int_{p_j}^{\min\{1+p_j-p_i,1\}} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} f(\theta_j - p_j + p_i, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j,$$

which is continuous in p.

Function $\frac{\partial \Psi^i_j}{\partial p_i}(p)$ indeed has a kink at each p with $p_i = p_j$ but is differentiable in p_i at every other p: For each p that $p_i < p_j$,

$$\frac{\partial^2 \Psi^i_j}{\partial p_i^2}(p) = \int_{p_j}^{1} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} (-1) \frac{\partial f}{\partial \theta_i}(\theta_j - p_j + p_i, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j;$$

For each p that $p_i > p_j$,

$$\frac{\partial^2 \Psi^i_j}{\partial p_i^2}(p) = \int_{p_j}^{1+p_j-p_i} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} (-1) \frac{\partial f}{\partial \theta_i}(\theta_j - p_j + p_i, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j$$
$$+ \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq 1 - p_i} f(1, 1 + p_j - p_i, \theta_{-(i,j)}) d\theta_{-(i,j)}$$

Therefore, $\frac{\partial^2 D_i}{\partial p_i^2}(p)$ only exists at p where $p_i \neq p_j$ for any j, and at such point p

$$\frac{\partial^2 D_i}{\partial p_i^2}(p) = \alpha^i(p) + \sum_{j : p_j < p_i} \beta^i_j(p),$$

where

$$\alpha^i(p) := \frac{\partial^2 \Psi^i_0}{\partial p_i^2}(p) + \sum_{j \neq i} \int_{p_j}^{\min\{1+p_j-p_i,1\}} \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq \theta_j - p_j} (-1) \frac{\partial f}{\partial \theta_i}(\theta_j - p_j + p_i, \theta_j, \theta_{-(i,j)}) d\theta_{-(i,j)} d\theta_j,$$

and

$$\beta_j^i(p) := \int_{\theta_{-(i,j)} - p_{-(i,j)} \leq 1-p_i} f(1, \min\{1 + p_j - p_i, 1\}, \theta_{-(i,j)}) d\theta_{-(i,j)}.$$

Note both $\alpha^i(p)$ and $\beta_j^i(p)$ are continuous in p. However $\frac{\partial^2 D_i}{\partial p_i^2}(p)$ is not continuous in p since the set $\{j : p_j < p_i\}$ changes with p.

Therefore, $\frac{\partial^2 \pi_i}{\partial p_i^2}(p)$ only exists at p where $p_i \neq p_j$ for any j, and at such point p

$$\frac{\partial^2 \pi_i}{\partial p_i^2}(p) = 2\frac{\partial D_i}{\partial p_i}(p) + p_i \frac{\partial^2 D_i}{\partial p_i^2}(p) = 2\frac{\partial D_i}{\partial p_i}(p) + p_i \left(\alpha^i(p) + \sum_{j: p_j < p_i} \beta_j^i(p) \right)$$

Step 2: I show there exists a $\delta > 0$ and a corresponding neighborhood of p^*—$B(p^*, \delta) := \{p \in [0,1]^n : |p_i - p_i^*| \leq \delta, \ \forall i\}$ such that for each i and at each $p \in B(p^*, \delta)$ where $\frac{\partial^2 \pi_i}{\partial p_i^2}(p)$ exists, $\frac{\partial^2 \pi_i}{\partial p_i^2}(p) < 0$.

Let $\bar{\delta}$ denote the smallest difference between any distinct $p_i^* \neq p_j^*$ (in case of $p_j^* = \bar{p}, \forall j$, let $\bar{\delta} := 1$):

$$\bar{\delta} := \min_{(i,j) \in N^2 : i \neq j, p_i^* \neq p_j^*} |p_i^* - p_j^*|.$$

Let $\delta < \frac{\bar{\delta}}{4}$. Then for each i and each $p \in B(p^*, \delta)$, $p_j < p_i$ only if $p_j^* \leq p_i^*$.

Define $\gamma^i(\cdot)$ with domain $B(p^*, \delta)$ as

$$\gamma^i(p) := 2\frac{\partial D_i}{\partial p_i}(p) + p_i \left(\alpha^i(p) + \sum_{j \neq i : p_j^* \leq p_i^*} \beta_j^i(p) \right)$$

Note $\gamma^i(p)$ is continuous in $B(p^*, \delta)$ and at each $p \in B(p^*, \delta)$ that $\frac{\partial^2 \pi_i}{\partial p_i^2}(p)$ exists,

$$\frac{\partial^2 \pi_i}{\partial p_i^2}(p) = 2\frac{\partial D_i}{\partial p_i}(p) + p_i \left(\alpha^i(p) + \sum_{j: p_j < p_i} \beta_j^i(p) \right) \leq \gamma^i(p),$$

because set $\{j : p_j < p_i\}$ is a subset of set $\{j : p_j^* \leq p_i^*\}$ and $\beta_j^i(p) \geq 0$.

Note

$$\gamma^i(p^*) = \lim_{p_i \to (p_i^*)^+} \frac{\partial^2 \pi_i}{\partial p_i^2}(p_i, p_{-i}^*) < 0$$

The inequality is due to similar argument in proof of Lemma 1 part (c):

$$\frac{\partial \pi_i}{\partial p_i}(p) = D_i(p) + p_i \frac{\partial D_i}{\partial p_i}(p) = D_i(p)\left(1 + p_i \frac{1}{D_i(p)}\frac{\partial D_i}{\partial p_i}(p)\right).$$

Therefore, at (p_i, p_i^*), where $p_i \in (p_i^*, p_i^* + \varepsilon)$ and ε is very small so that $p_i \neq p_j^*$ for any j,

$$\frac{\partial^2 \pi_i}{\partial p_i^2}(p_i, p_i^*) = \frac{\partial D_i}{\partial p_i}(p_i, p_i^*)\left(1 + p_i \frac{1}{D_i(p_i, p_i^*)}\frac{\partial D_i}{\partial p_i}(p_i, p_i^*)\right) + D_i(p_i, p_i^*)\frac{\partial}{\partial p_i}\left(1 + p_i \frac{1}{D_i(p_i, p_i^*)}\frac{\partial D_i}{\partial p_i}(p_i, p_i^*)\right).$$

Note,

$$\left(1 + p_i \frac{1}{D_i(p_i, p_i^*)}\frac{\partial D_i}{\partial p_i}(p_i, p_i^*)\right) = \frac{1}{D_i(p_i, p_i^*)}\frac{\partial \pi_i}{\partial p_i}(p_i, p_i^*) \to 0, p_i \to p_i^*$$

$$\frac{\partial}{\partial p_i}\left(\frac{1}{D_i(p_i, p_i^*)}\frac{\partial D_i}{\partial p_i}(p_i, p_i^*)\right) = \frac{\partial^2 \log(D_i)}{\partial p_i^2}(p_i, p_i^*).$$

Therefore,

$$\lim_{p_i \to (p_i^*)^+} \frac{\partial^2 \pi_i}{\partial p_i^2}(p_i, p_{-i}^*) = D_i(p^*)\left(\frac{1}{D_i(p^*)}\underbrace{\frac{\partial D_i}{\partial p_i}(p^*)}_{<0} + p_i^* \lim_{p_i \to (p_i^*)^+} \underbrace{\frac{\partial^2 \log(D_i)}{\partial p_i^2}(p_i, p_i^*)}_{\leq 0 \text{ as } D_i \text{ is log-concave in } p_i}\right) < 0.$$

Hence, by choosing δ to be sufficiently small, I have

$$\frac{\partial^2 \pi_i}{\partial p_i^2}(p) \leq \gamma^i(p) < 0,$$

for each $p \in B(p^*, \delta)$ that $\frac{\partial^2 \pi_i}{\partial p_i^2}(p)$ exists.

B.5 Proof of Lemma 5

Note

$$\frac{\partial \pi_i}{\partial p_i}(p) = D_i(p) + p_i \frac{\partial D_i}{\partial p_i}(p).$$

From proof of Lemma 4,

$$D_i(p) = \Psi_0^i(p) + \sum_{j \neq i} \Psi_j^i(p),$$

$$\Psi_0^i(p) = \int_{p_i}^{1} \int_{\theta_{-i} \leq p_{-i}} f(\theta_i, \theta_{-i}) d\theta_{-i} d\theta_i$$

For $j \in N, j \neq i$,

$$\Psi_j^i(p) = \int_{p_j}^{\min\{1+p_j-p_i,1\}} \int_{\theta_{-(i,j)}-p_{-(i,j)} \leq \theta_j - p_j} \int_{\theta_j-p_j+p_i}^{1} f(\theta_i, \theta_j, \theta_{-(i,j)}) d\theta_i d\theta_{-(i,j)} d\theta_j$$

By Lemma 4, given p_{-i}, $\frac{\partial \pi_i}{\partial p_i}(p)$ is a continuous in p_1. I am interested in whether $\frac{\partial \pi_i}{\partial p_i}(p_1, p_{-1}(\varepsilon))$ is continuously differentiable in p_1 for $p_1 \in (p_1^* - \varepsilon, p_1^*]$. Note $p_{-1}(\varepsilon) = p_{-1}^*$.

If $i = 1$, since $p_1 \leq p_j^*$,

$$\Psi_j^1(p_1, p_{-1}^*) = \int_{p_j^*}^{1} \int_{\theta_{-(1,j)}-p^*_{-(1,j)} \leq \theta_j - p_j^*} \int_{\theta_j-p_j^*+p_1}^{1} f(\theta_1, \theta_j, \theta_{-(1,j)}) d\theta_1 d\theta_{-(1,j)} d\theta_j$$

Therefore, $\frac{\partial \pi_1}{\partial p_1}(p_1, p_{-1}^*)$ is continuously differentiable in p_1 over $(p_1^* - \varepsilon, p_1^*]$ since I assumed f is continuously differentiable.

Now consider $i \neq 1$.

Note,

$$D_i(p_1, p_{-1}^*) = \int_{p_i^*}^{1} \int_{\theta_{-(i,1))}-p^*_{-(i,1)} \leq \theta_i - p_i^*} \int_{0}^{\theta_i - p_i^* + p_1} f(\theta_i, \theta_1, \theta_{-(i,1)}) d\theta_1 d\theta_{-(i,1)} d\theta_i$$

which is is continuously differentiable in p_1 over $(p_1^* - \varepsilon, p_1^*]$.

Note

$$\frac{\partial \Psi_0^i}{\partial p_i}(p) = -\int_{\theta_{-(i,1))} \leq p_{-(i,1))}} \int_{0}^{p_1} f(p_i, \theta_1, \theta_{-(i,1))}) d\theta_1 d\theta_{-(i,1))},$$

which is continuously differentiable in p_1 over $(p_1^* - \varepsilon, p_1^*]$.

Note

$$\frac{\partial \Psi_j^i}{\partial p_i}(p) = -\int_{p_j}^{\min\{1+p_j-p_i,1\}} \int_{\theta_{-(i,j)}-p_{-(i,j)}\le\theta_j-p_j} f(\theta_j-p_j+p_i,\theta_j,\theta_{-(i,j)})d\theta_{-(i,j)}d\theta_j,$$

For $j\neq 1$,

$$\frac{\partial \Psi_j^i}{\partial p_i}(p_1,p_{-1}^*) = -\int_{p_j^*}^{\min\{1+p_j^*-p_i^*,1\}} \int_{\theta_{-(i,j,1)}-p^*_{-(i,j,1)}\le\theta_j-p_j^*} \int_0^{\theta_j-p_j^*+p_1} f(\theta_j-p_j^*+p_i^*,\theta_j,\theta_{-(i,j,1)},\theta_1)d\theta_1 d\theta_{-(i,j,1)}d\theta_j$$

which is continuously differentiable in p_1 over $(p_1^*-\varepsilon,p_1^*]$.

For $j=1$,

$$\frac{\partial \Psi_1^i}{\partial p_i}(p_1,p_{-1}^*) = -\int_{p_1}^{1+p_1-p_i^*} \int_{\theta_{-(i,1)}-p^*_{-(i,1)}\le\theta_1-p_1} f(\theta_1-p_1+p_i^*,\theta_1,\theta_{-(i,1)})d\theta_{-(i,1)}d\theta_1,$$

$$= -\int_{p_1}^{1+p_1-p_i^*} \sum_{k\neq i,1} \int_0^{\min\{\theta_1-p_1+p_k^*,1\}} \int_{\theta_{-(i,1,k)}-p^*_{-(i,1,k)}\le\theta_k-p_k^*} f(\theta_1-p_1+p_i^*,\theta_1,\theta_{-(i,1,k)},\theta_k)d\theta_{-(i,1,k)}d\theta_k d\theta_1$$

$$= -\sum_{k\neq i,1} \underbrace{\int_{p_1}^{1+p_1-p_i^*} \int_0^{\min\{\theta_1-p_1+p_k^*,1\}} \int_{\theta_{-(i,1,k)}-p^*_{-(i,1,k)}\le\theta_k-p_k^*} f(\theta_1-p_1+p_i^*,\theta_1,\theta_{-(i,1,k)},\theta_k)d\theta_{-(i,1,k)}d\theta_k d\theta_1}_{:=I_k^i(p_1,p_{-1}^*)}$$

For k that $p_k^*\le p_i^*$,

$$I_k^i(p_1,p_{-1}^*) = \int_{p_1}^{1+p_1-p_i^*} \int_0^{\theta_1-p_1+p_k^*} \int_{\theta_{-(i,1,k)}-p^*_{-(i,1,k)}\le\theta_k-p_k^*} f(\theta_1-p_1+p_i^*,\theta_1,\theta_{-(i,1,k)},\theta_k)d\theta_{-(i,1,k)}d\theta_k d\theta_1,$$

which is continuously differentiable in p_1 over $(p_1^*-\varepsilon,p_1^*]$.

For k that $p_k^*>p_i^*$,

$$I_k^i(p_1,p_{-1}^*) = \int_{p_1}^{1+p_1-p_k^*} \int_0^{\theta_1-p_1+p_k^*} \int_{\theta_{-(i,1,k)}-p^*_{-(i,1,k)}\le\theta_k-p_k^*} f(\theta_1-p_1+p_i^*,\theta_1,\theta_{-(i,1,k)},\theta_k)d\theta_{-(i,1,k)}d\theta_k d\theta_1$$

$$+\int_{1+p_1-p_k^*}^{1+p_1-p_i^*} \int_0^1 \int_{\theta_{-(i,1,k)}-p^*_{-(i,1,k)}\le\theta_k-p_k^*} f(\theta_1-p_1+p_i^*,\theta_1,\theta_{-(i,1,k)},\theta_k)d\theta_{-(i,1,k)}d\theta_k d\theta_1,$$

which is continuously differentiable in p_1 over $(p_1^* - \varepsilon, p_1^*]$. Therefore, $\frac{\partial \Psi_1^i}{\partial p_i}(p_1, p_{-1}^*)$ is continuously differentiable in p_1 over $(p_1^* - \varepsilon, p_1^*]$.

To conclude, $\frac{\partial \pi_i}{\partial p_i}(p_1, p_{-1}(\varepsilon))$ is continuously differentiable in p_1 over $(p_1^* - \varepsilon, p_1^*]$.